Section 1.2. Sets

Note. Ironically, we never formally define a set, but leave it as the intuitive idea of a "collection" of objects called "elements." In this section we present several ideas from "naive" set theory; for a more treatment, see my online notes for Introduction to Set Theory. We also define some functions on sets. Such functions will play an important role in the future. We assume a familiarity with sets of real numbers at the level of Calculus 1 where sets are described using interval notation or inequalities:

$$\{x \in \mathbb{R} \mid 1 \leq 2\} = \{x \in \mathbb{R} \mid x \in [1, 2)\}.$$

Definition. A set C is *countable* if it is either finite or if it has "as many elements" as there are natural numbers (that is, there is a one to one and onto function from C to $\mathbb{N}$).

Note. Some infinite countable sets are $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{Q} \times \mathbb{Q}$. Surprisingly, there are uncountable sets! The interval $(0, 1)$ is not countable (see Theorem 1-20 of my Analysis 1 notes for 1.3. The Completeness Axiom).

Definition 1.2.2. The *complement* of an event A is the set of all elements in $\mathcal{C}$ which are not in A, denoted A^c. That is, $A^c = \{x \in \mathcal{C} \mid x \notin A\}$.

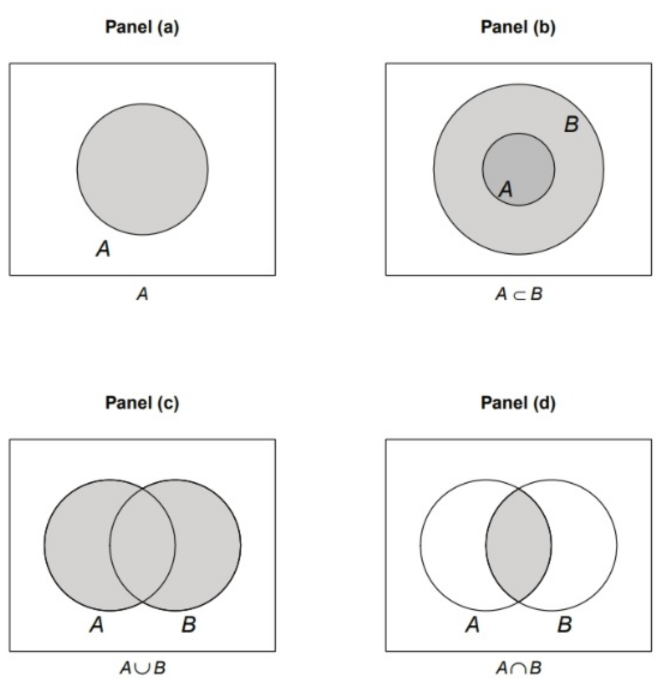
Definition 1.2.2. If each element of set A is also an element of set B, then set A is a *subset* of set B, denoted $A \subset B$ or $B \supset A$. If $A \subset B$ and $B \subset A$ then sets A and B are *equal*, denoted $A = B$.

Definition 1.2.3. Let A and B be events. The *union* of A and B is the set of all elements that are in A or in B or in both A and B, denoted $A \cup B$.

Definition 1.2.4. Let A and B be events. The *intersection* of A and B is the set of all elements that are in both A and B, denoted $A \cap B$.

Definition 1.2.5. Let A and B be events. Then A and B are *disjoint* if $A \cap B = \varnothing$, where $\varnothing$ denotes the empty set. If A and B are disjoint then $A \cup B$ is called the *disjoint union*, which we denote as $A \mathbin{\dot\cup} B$ (though the text does not use this notation).

Note. The text illustrates the definitions above definitions with Venn diagrams as follows:



Theorem 1.2.A. For any sets (events) A, B, and C we have

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \text{ and } A \cup (C \cap C) = (A \cup B) \cap (A \cup C).$$

These are the *distributive laws.*

Theorem 1.2.B. De Morgan's Laws. For any two sets (events) A and B, we have

$$(A \cap B)^c = A^c \cup B^c \text{ and } (A \cup B)^c = A^c \cap B^c.$$

Definition. For sets $A_1, A_2, \ldots, A_n$ define the *union* and *intersection*, respectively, as

$$A_1 \cup A_2 \cup \cdots \cup A_n = \cup_{i=1}^n A_i = \{x \mid x \in A_i \mid \text{ for some } i \in \{1, 2, \ldots, n\}\},$$

$$A_1 \cap A_2 \cap \cdots \cap A_n = \cap_{i=1}^n A_i = \{x \mid x \in A_i \mid \text{ for all } i \in \{1, 2, \ldots, n\}\}.$$

For a sequence of sets $A_1, A_2, \ldots$ define the *union* and *intersection*, respectively, as

$$A_1 \cup A_2 \cup \cdots = \cup_{i=1}^\infty A_i = \{x \mid x \in A_i \mid \text{ for some } i \in \mathbb{N}\},$$

$$A_1 \cap A_2 \cap \cdots = \cap_{i=1}^\infty A_i = \{x \mid x \in A_i \mid \text{ for all } i \in \mathbb{N}\}.$$

More generally, for any collection of set A_i where $i \in I$ and I is some indexing set (finite, countable, or uncountable), define the *union* and *intersection*, respectively, as

$$\cup_{i \in I} A_i = \{x \mid x \in A_i \mid \text{ for some } i \in I\},$$

$$\cap_{i \in I} A_i = \{x \mid x \in A_i \mid \text{ for all } i \in I\}.$$

Note. De Morgan's Laws hold for finite, countable, and arbitrary unions.

Example 1.2.4. Suppose $\mathcal{C}$ is the interval of real numbers $(0, 5)$. Define $C_n = (1 - 1/n, 2 + 1/n)$ and $D_n = (1/n, 3 - 1/n)$ for $n \in \mathbb{N}$. Then $\cup_{n=1}^{\infty} C_n = (0, 3)$, $\cap_{n=1}^{\infty} C_n = [1, 2]$, $\cup_{n=1}^{\infty} D_n = (0, 3)$, and $\cap_{n=1}^{\infty} D_n = (1, 2)$. This example motivates the following definition.

Definition. A sequence of sets $\{A_n\}$ is *monotone nondecreasing* (also called "monotone increasing") if $A_n \subset A_{n+1}$ for $n \in \mathbb{N}$. The sequence is *monotone nonincreasing* (also called "monotone decreasing") is $A_n \supset A_{n+1}$ for $n \in \mathbb{N}$. For monotone nondecreasing sequence $\{A_n\}$ define the *limit* $\lim_{n\to\infty} A_n = \cup_{n=1}^{\infty} A_n$. For monotone increasing sequence $\{A_n\}$ define the *limit* $\lim_{n\to\infty} A_n = \cap_{n=1}^{\infty} A_n$.

Definition. A function that maps sets into the real numbers is called a *set function.*

Example. If $f(x) = e^{-x^2}$ then we can define set function F on closed intervals of real numbers as $F([a, b]) = \int_a^b e^{-x^2}\, dx$.

Note. We will often deal with definite integrals. For $A \subset \mathbb{R}$ and f a real valued function, we denote the integral of f over A as $\int_A f(x)\, dx$, though we may have legitimate concerns about the integral existing (or being finite). If $A \subset \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ then we denote an integral of $g : \mathbb{R}^2 \to \mathbb{R}$ as $\int \int_A g(x, y)\, dx\, dy$.

Example 1.2.7. Let $\mathcal{C} = \mathbb{N}$ and let $A \subset \mathcal{C}$. Define the set function

$$Q(A) = \sum_{n \in A} \left(\frac{2}{3}\right)^n.$$

Then

$$Q(\mathcal{C}) = \sum_{n \in \mathbb{N}} \left(\frac{2}{3}\right)^n = \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{1 - 2/3} = 3.$$

Let B be the set of off positive integers. Then

$$Q(B) = \sum_{n \in B} \left(\frac{2}{3}\right)^n = \sum_{i=0} \left(\frac{2}{3}\right)^{2n+1} = \frac{2}{3} \sum_{i=0}^{\infty} \left(\frac{4}{9}\right)^n = \frac{2}{3} \frac{1}{1 - 4/9} = \frac{6}{5}. \qquad \square$$

Example 1.2.9. Let $\mathcal{C} = \mathbb{R}^n$. for $A \subset \mathcal{C}$ define the set function $Q(A) = \int\int\cdots\int_A dx_1\,dx_2\cdots dx_n$, provided the integral exists (notice this is an integral of the function 1 so we expect the quantity $Q(A)$ to be the n-dimensional volume of A). Let

$$A = \{(x_1, x_2, \ldots, x_n) \mid - \leq x_1 \leq x_2 \text{ and } 0 \leq x_i \leq 1 \text{ for } i \in \{2, 3, 4, \ldots, n\}\}.$$

Then

$$Q(A) = \int_0^1 \int_0^1 \cdots \int_0^1 \int_0^{x_2} dx_1\,dx_2\cdots dx_n = (1) \int_0^1 \left(\int_0^{x_2} dx_1 \right) dx_2$$

$$= \int_0^1 \left(x_1 |_{x_1=0}^{x_1=x_2} \right) dx_2 = \int_0^1 (x_2 - 0)\,dx_2 = \frac{1}{2} x_2^2 |_{x_2=0}^{x_2=1} = \frac{1}{2}.$$

If $B = \{(x_1, x_2, \ldots, x_n) \mid 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq 1\}$ then

$$\begin{aligned} Q(B) &= \int_0^1 \int_0^{x_n} \cdots \int_0^{x_3} \int_0^{x_2} dx_1\,dx_2\cdots dx_{n-1}\,dx_n \\ &= \int_0^1 \int_0^{x_n} \cdots \int_0^{x_3} \left(x_1 |_{x_1=0}^{x_1=x_2} \right) dx_2\cdots dx_{n-1}\,dx_n \end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \int_0^{x_n} \cdots \int_0^{x_3} x_2 \, dx_2 \cdots dx_{n-1} \, dx_n \\
&= \int_0^1 \int_0^{x_n} \cdots \int_0^{x_4} \frac{1}{2} x_3^2 \, dx_3 \cdots dx_{n-1} \, dx_n \\
&= \int_0^1 \int_0^{x_n} \cdots \int_0^{x_5} \frac{1}{(1)(2)(3)} x_4^3 \, dx_4 \cdots dx_{n-1} \, dx_n \\
&= \cdots = \int_0^1 \frac{1}{(n-1)!} x_n^{n-1} \, dx_n = \frac{1}{n!} x_n^n |_{x_n=0}^{x_x=1} = \frac{1}{n!}.
\end{aligned}
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