Section 1.2. Sets

Note. Ironically, we never formally define a set, but leave it as the intuitive idea of a “collection” of objects called “elements.” In this section we present several ideas from “naive” set theory; for a more treatment, see my online notes for Introduction to Set Theory. We also define some functions on sets. Such functions will play an important role in the future. We assume a familiarity with sets of real numbers at the level of Calculus 1 where sets are described using interval notation or inequalities:

\[ \{ x \in \mathbb{R} \mid 1 \leq x \} = \{ x \in \mathbb{R} \mid x \in [1, 2) \} . \]

Definition. A set \( C \) is countable if it is either finite or if it has “as many elements” as there are natural numbers (that is, there is a one to one and onto function from \( C \) to \( \mathbb{N} \)).

Note. Some infinite countable sets are \( \mathbb{N} \), \( \mathbb{Z} \), \( \mathbb{Q} \), and \( \mathbb{Q} \times \mathbb{Q} \). Surprisingly, there are uncountable sets! The interval \( (0, 1) \) is not countable (see Theorem 1-20 of my Analysis 1 notes for 1.3. The Completeness Axiom).

Definition 1.2.2. The complement of an event \( A \) is the set of all elements in \( \mathcal{C} \) which are not in \( A \), denoted \( A^c \). That is, \( A^c = \{ x \in \mathcal{C} \mid x \not\in A \} \).

Definition 1.2.2. If each element of set \( A \) is also an element of set \( B \), then set \( A \) is a subset of set \( B \), denoted \( A \subseteq B \) or \( B \supseteq A \). If \( A \subseteq B \) and \( B \subseteq A \) then sets \( A \) and \( B \) are equal, denoted \( A = B \).
**Definition 1.2.3.** Let $A$ and $B$ be events. The *union* of $A$ and $B$ is the set of all elements that are in $A$ or in $B$ or in both $A$ and $B$, denoted $A \cup B$.

**Definition 1.2.4.** Let $A$ and $B$ be events. The *intersection* of $A$ and $B$ is the set of all elements that are in both $A$ and $B$, denoted $A \cap B$.

**Definition 1.2.5.** Let $A$ and $B$ be events. Then $A$ and $B$ are *disjoint* if $A \cap B = \emptyset$, where $\emptyset$ denotes the empty set. If $A$ and $B$ are disjoint then $A \cup B$ is called the *disjoint union*, which we denote as $A \cup B$ (though the text does not use this notation).

**Note.** The text illustrates the definitions above definitions with Venn diagrams as follows:
Theorem 1.2.A. For any sets (events) $A$, $B$, and $C$ we have

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \text{ and } A \cup (C \cap C) = (A \cup B) \cap (A \cup C).$$

These are the distributive laws.

Theorem 1.2.B. De Morgan’s Laws. For any two sets (events) $A$ and $B$, we have

$$(A \cap B)^c = A^c \cup B^c \text{ and } (A \cup B)^c = A^c \cap B^c.$$

Definition. For sets $A_1, A_2, \ldots, A_n$ define the union and intersection, respectively, as

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i = \{ x \mid x \in A_i \text{ for some } i \in \{1, 2, \ldots, n\} \},$$

$$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i = \{ x \mid x \in A_i \text{ for all } i \in \{1, 2, \ldots, n\} \}.$$

For a sequence of sets $A_1, A_2, \ldots$ define the union and intersection, respectively, as

$$A_1 \cup A_2 \cup \cdots = \bigcup_{i=1}^{\infty} A_i = \{ x \mid x \in A_i \text{ for some } i \in \mathbb{N} \},$$

$$A_1 \cap A_2 \cap \cdots = \bigcap_{i=1}^{\infty} A_i = \{ x \mid x \in A_i \text{ for all } i \in \mathbb{N} \}.$$

More generally, for any collection of set $A_i$ where $i \in I$ and $I$ is some indexing set (finite, countable, or uncountable), define the union and intersection, respectively, as

$$\bigcup_{i \in I} A_i = \{ x \mid x \in A_i \text{ for some } i \in I \},$$

$$\bigcap_{i \in I} A_i = \{ x \mid x \in A_i \text{ for all } i \in I \}.$$
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Note. De Morgan’s Laws hold for finite, countable, and arbitrary unions.

Example 1.2.4. Suppose $\mathcal{C}$ is the interval of real numbers $(0, 5)$. Define $C_n = (1 - 1/n, 2 + 1/n)$ and $D_n = (1/n, 3 - 1/n)$ for $n \in \mathbb{N}$. Then $\bigcup_{n=1}^\infty C_n = (0, 3)$, $\bigcap_{n=1}^\infty C_n = [1, 2]$, $\bigcup_{n=1}^\infty D_n = (0, 3)$, and $\bigcap_{n=1}^\infty D_n = (1, 2)$. This example motivates the following definition.

Definition. A sequence of sets $\{A_n\}$ is monotone nondecreasing (also called “monotone increasing”) if $A_n \subset A_{n+1}$ for $n \in \mathbb{N}$. The sequence is monotone nonincreasing (also called “monotone decreasing”) is $A_n \supset A_{n+1}$ for $n \in \mathbb{N}$. For monotone nondecreasing sequence $\{A_n\}$ define the limit $\lim_{n \to \infty} A_n = \bigcup_{n=1}^\infty A_n$. For monotone increasing sequence $\{A_n\}$ define the limit $\lim_{n \to \infty} A_n = \bigcap_{n=1}^\infty A_n$.

Definition. A function that maps sets into the real numbers is called a set function.

Example. If $f(x) = e^{-x^2}$ then we can define set function $F$ on closed intervals of real numbers as $F([a, b]) = \int_a^b e^{-x^2} \, dx$.

Note. We will often deal with definite integrals. For $A \subset \mathbb{R}$ and $f$ a real valued function, we denote the integral of $f$ over $A$ as $\int_A f(x) \, dx$, though we may have legitimate concerns about the integral existing (or being finite). If $A \subset \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ then we denote an integral of $g : \mathbb{R}^2 \to \mathbb{R}$ as $\int \int_A g(x, y) \, dx \, dy$. 


Example 1.2.7. Let $C = \mathbb{N}$ and let $A \subset C$. Define the set function

$$Q(A) = \sum_{n \in A} \left( \frac{2}{3} \right)^n.$$ 

Then

$$Q(C) = \sum_{n \in \mathbb{N}} \left( \frac{2}{3} \right)^n = \sum_{i=0}^{\infty} \left( \frac{2}{3} \right)^n = \frac{1}{1 - 2/3} = 3.$$ 

Let $B$ be the set of off positive integers. Then

$$Q(B) = \sum_{n \in B} \left( \frac{2}{3} \right)^n = \sum_{i=0}^{\infty} \left( \frac{2}{3} \right)^{2n+1} = \frac{2}{3} \sum_{i=0}^{\infty} \left( \frac{4}{9} \right)^n = \frac{2}{3} \frac{1}{1 - 4/9} = \frac{6}{5}. \quad \square$$

Example 1.2.9. Let $C = \mathbb{R}^n$. For $A \subset C$ define the set function $Q(A) = \int \cdots \int_A dx_1 dx_2 \cdots dx_n$, provided the integral exists (notice this is an integral of the function $1$ so we expect the quantity $Q(A)$ to be the $n$-dimensional volume of $A$). Let

$$A = \{(x_1, x_2, \ldots, x_n) \mid -1 \leq x_1 \leq x_2 \text{ and } 0 \leq x_i \leq 1 \text{ for } i \in \{2, 3, 4, \ldots, n\}\}.$$ 

Then

$$Q(A) = \int_0^1 \int_0^1 \cdots \int_0^1 \int_0^{x_2} dx_1 dx_2 \cdots dx_n = (1) \int_0^1 \left( \int_0^{x_2} dx_1 \right) dx_2$$

$$= \int_0^1 (x_1 |_{x_1=0}) dx_2 = \int_0^1 (x_2 - 0) dx_2 = \frac{1}{2} x_2 |_{x_2=0} = \frac{1}{2}.$$ 

If $B = \{(x_1, x_2, \ldots, x_n) \mid 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq 1\}$ then

$$Q(B) = \int_0^1 \int_0^{x_n} \cdots \int_0^{x_3} \int_0^{x_2} dx_1 dx_2 \cdots dx_{n-1} dx_n$$

$$= \int_0^1 \int_0^{x_n} \cdots \int_0^{x_3} (x_1 |_{x_1=0}) dx_2 \cdots dx_{n-1} dx_n$$
$$= \int_0^1 \int_0^{x_n} \cdots \int_0^{x_3} x_2 \, dx_2 \cdots dx_{n-1} \, dx_n$$

$$= \int_0^1 \int_0^{x_n} \cdots \int_0^{x_4} \frac{1}{2} x_3^2 \, dx_3 \cdots dx_{n-1} \, dx_n$$

$$= \int_0^1 \int_0^{x_n} \cdots \int_0^{x_5} \frac{1}{(1)(2)(3)} x_4^3 \, dx_4 \cdots dx_{n-1} \, dx_n$$

$$= \cdots = \int_0^1 \frac{1}{(n-1)!} x_{n-1} \, dx_n = \frac{1}{n!} x_n |_{x_n=0}^{x_n=1} = \frac{1}{n!}.$$