# Section 1.6. Discrete Random Variables

**Note.** We now formally define some of the ideas illustrated in the previous section.

**Definition 1.6.1.** A random variable is a *discrete random variable* if the space (its range) is either finite or countable.

**Example 1.6.1.** Suppose a fair coin is flipped an infinite number of times. Let the random variable $X$ equal the number of flips needed to obtain the first head ($H$). Then the sample space $\mathcal{C}$ consists of all sequences of $H$'s and $T$'s (an uncountable sample space). The space is $\mathcal{D} = \{1, 2, \ldots\} = \mathbb{N}$, so $X$ is a discrete random variable. Notice that $X = 1$ corresponds to the events $c \in \mathcal{C}$ such that $X(c) = 1$, so that this includes all sequences of events that start with $H$ (an uncountable collection). For $x \in \mathbb{N}$ we have $P(X = x) = (1/2)^x$ since this requires a sequence of $(x-1)$ $T$'s followed by a $H$. Each such outcome has probability 1/2 so the value of $P(X = x)$ follows. The probability that $X$ is odd is

$$P(X \in \{1, 3, 5, \ldots\}) = \sum_{x=1}^{\infty} \left(\frac{1}{2}\right)^{2x-1} = 2\sum_{x=1}^{\infty} \left(\frac{1}{4}\right)^{x} = 2\frac{1/4}{1-1/4} = \frac{2}{3}.$$

Notice the similarity of this to Exercise 1.4.18.

**Note.** Notice that each element of the sample space in Example 1.6.1, that is each infinite sequence of $T$'s and $H$'s, has probability 0. This gives an example of an experiment where an event is *possible* yet it has probability 0 (consider the outcome $TTT\cdots$, for example, or any outcome for that matter).

**Definition 1.6.2.** Let $X$ be a discrete random variable with space $\mathcal{D}$. The *probability mass function* of $X$ is $p_X(x) = P(X = x)$ for $x \in \mathcal{D}$. The *support* of discrete random variable $X$, denoted $\mathcal{S}$, is the set of points in the space ("range") of $X$ which has positive probability: $\mathcal{S} = \{x \in \mathcal{D} \mid p_X(x) = P(X = x) > 0\}$.

**Note.** By Theorem I.5.3, $P(X = x) = F_X(x) - F_X(x^-)$ where $F_X(x^-) = \lim_{z \to x^-} F_X(z)$, so $P(X = x) = 0$ if and only $F_X$ is continuous at $x$. So the support of discrete random variable $X$ is the set of points of discontinuity of the cumulative distribution function $F_X$.

**Note.** The following can be shown "in a more advanced class" (see Hogg, McKean, and Craig page 46).

**Theorem 1.6.A.** Let $\mathcal{D}$ be a finite or countable set of real numbers. Then function $p_X : \mathcal{D} \to \mathbb{R}$ is a probability mass function for some discrete random variable $X$ is and only if

**(i)** $0 \le p_X(x) \le 1$ for all $x \in \mathcal{D}$, and

**(ii)** $\sum_{x \in \mathcal{D}} p_X(x) = 1$.

**Example 1.6.2.** A lot of 100 fuses is inspected by the following process. Five of these fuses are chosen at random and tested; if all five "blow" at the correct amperage, then the lot is accepted. Let $X$ be the number of defective fuses among the five that are inspected. Then $X$ is a discrete random variable with space $\mathcal{D} = \{0, 1, 2, 3, 4, 5\}$. The probability mass function of $X$ is

$$p_X(x) = \begin{cases} \frac{\binom{20}{x}\binom{80}{5-x}}{\binom{100}{5}} & \text{for } x = 0, 1, 2, 3, 4, 5 \\ 0 & \text{elsewhere.} \end{cases}$$

This is a particular example of a hypergeometric distribution, which we will explore in some detail in Chapter 3.

**Note/Definition.** Suppose we have a random variable $X$ with distribution $p_X$. If for some function $g$ we have $Y = g(X)$ then $g$ is called a *transformation*. If $X$ is a discrete random variable and the space $X$ is $\mathcal{D}_X$, then the space of $Y$ is $\mathcal{D}_Y = \{g(x) \mid x \in \mathcal{D}_X\}$. If function $g^{-1}$ exists (i.e., if $g$ is one to one) then

$$p_Y(y) = P(Y = y) = P(g(X) = y) = P(X = g^{-1}(y)) = p_X(g^{-1}(y).$$

**Example 1.6.4.** Let discrete random variable $X$ have probability mass function

$$p_X(x) = \begin{cases} \frac{3!}{x!(3-x)!}\left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{3-x} & \text{for } x = 0, 1, 2, 3 \\ 0 & \text{elsewhere.} \end{cases}$$

Let $Y$ be the discrete random variable defined as $Y = X^2$. With $y = g(x) = x^2$ as the transformation we have $\mathcal{D}_X = \{0, 1, 2, 3\}$ and $\mathcal{D}_Y = \{y = g(x) = x^2 \mid x \in \mathcal{D}_X\} = \{0, 1, 4, 9\}$. Since $g$ is one to one on $\mathcal{D}_X$ then we have the relationship $x = \sqrt{y} = g^{-1}(y)$ for $y \in \mathcal{D}_Y$ and so

$$p_Y(y) = p_X(\sqrt{y}) = \begin{cases} \frac{3!}{(\sqrt{y})!(3-\sqrt{y})!}\left(\frac{2}{3}\right)^{\sqrt{y}} \left(\frac{1}{3}\right)^{3-\sqrt{y}} & \text{for } y = 0, 1, 4, 9 \\ 0 & \text{elsewhere.} \end{cases}$$

**Example.** Consider again the random variable of Example 1.6.1 which is equal to the number of flips of a fair coin needed to obtain the first head ($H$). Define the new discrete random variable $Z = (X-2)^2$ so that the transformation is $g(x) = (x-2)^2$. Since $\mathcal{D}_X = \{1, 2, 3, \ldots\}$ then $\mathcal{D}_Z = \{g(x) \mid x \in \mathcal{D}_X\} = \{0, 1, 4, 9, 16, \ldots\}$ but $g$ is not one to one on all of $\mathcal{D}_X$. Now $Z = 0$ if and only if $X = 2$, and $Z = 1$ if and only if $X = 1$ or $X = 3$. For the other values of $Z$ (i.e., for $z \geq 4$) we have $x = \sqrt{z} + 2$. So we have the probability mass function for $Z$ as

$$p_Z(z) = \begin{cases} p_X(2) = (1/2)^2 = 1/4 & \text{for } z = 0 \\ p_X(1) + p_X(3) = 1/2 + 1/8 = 5/8 & \text{for } z = 1 \\ p_X(\sqrt{z}+2) = (1/2)^{\sqrt{z}+2} & \text{for } z = 4, 9, 16, \ldots. \end{cases}$$

Notice that

$$\sum_{z \in \mathcal{D}_Z} p_Z(z) = p_Z(0) + p_Z(1) + \sum_{z \in \{4,9,16,\ldots\}} p_Z(\sqrt{z}+2) = \frac{1}{4} + \frac{5}{8} + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{\sqrt{n^2}+2}$$

$$= \frac{7}{8} + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n+2} = \frac{7}{8} + \frac{1}{4}\sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^{n}$$

$$= \frac{7}{8} + \frac{1}{4}\left(\frac{1/4}{1-1/2}\right) = \frac{7}{8} + \frac{1}{8} = 1$$

(this last observation is Exercise 1.6.11). So by Theorem 1.6.A, $p_Z$ actually is a probability mass function.

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