Section 1.8. Expectation of a Random Variables

Note. We now define the expectation (or “expected value”) of both a discrete and continuous random variable.

Definition 1.8.1. Let $X$ be a random variable. If $X$ is a continuous random variable with probability density function $f$ and if $\int_{-\infty}^{\infty} |x| f(x) \, dx < \infty$, then the expectation of $X$ (or expected value or mean) is $E[X] = \int_{-\infty}^{\infty} x f(x) \, dx$. If $X$ is a discrete random variable with probability mass function $p$ and $\sum_{x} x |p(x)| < \infty$, then the expectation of $X$ (or expected value or mean) is $E[X] = \sum_{x} x p(x)$. Sometimes (especially when using the term “mean”) we denote $\mu = E(X)$.

Note. The hypothesis $\int_{-\infty}^{\infty} |x| f(x) \, dx < \infty$ is a hypothesis that function $xf(x)$ is “absolutely integrable.” If a function is absolutely integrable then it is integrable (that is, if $\int_{-\infty}^{\infty} |f(x)| \, dx < \infty$ then $\int_{-\infty}^{\infty} f(x) \, dx < \infty$). This follows from Corollary 6-15(b) of my Analysis 1 (MATH 4217/5217) notes on 6-2. Some Properties and Applications of the Riemann Integral (this holds for Lebesgue integrals; see Proposition 4.16. The Integral Comparison Test from my Real Analysis 1 [MATH 5201] on 4.4. The General Lebesgue Integral). So the hypothesis guarantees that the integral $\int_{-\infty}^{\infty} |x| f(x) \, dx$ exists as a finite value (recall that probability density function satisfies $|f(x)| = f(x) \geq 0$ for all $x \in \mathbb{R}$). The hypothesis $\sum_{x} |x| p(x) < \infty$ is a hypothesis of absolute summability of the series. Since the absolute convergence of a series implies the convergence of the series (see Theorem 10. The Absolute Convergence Test from my Calculus 2 [MATH 1920] notes on 10.6. Alternating Se-
ries, Absolute and Conditional Convergence) . . . this also guarantees that the series
can be rearranged (see Theorem 17. The Rearrangement Theorem for Absolutely
Convergent Series in the same section of notes) so that the ambiguity of “∑ₙ” is
irrelevant (i.e., the terms can be summed in any order).

**Example.** If a fair 6-sided die is rolled and the random variable \( X \) is the outcome
then the expected outcome is

\[
E(X) = \frac{1}{6}(1) + \frac{1}{6}(2) + \frac{1}{6}(3) + \frac{1}{6}(4) + \frac{1}{6}(5) + \frac{1}{6}(6) = \frac{7}{2}.
\]

**Note/Definition.** A game of chance is a *fair game* is the expected gain (that is,
the payoff minus the cost of playing) is 0.

**Note.** If we transform a random variable \( X \) to get \( Y = g(X) \) then we can use the
following theorem to compute \( E[Y] \).

**Theorem 1.8.1.** Let \( X \) be a random variable and let \( Y = g(X) \) for some function
\( g \).

(a) Suppose \( X \) is continuous with probability density function \( f_X(x) \). If

\[
\int_{-\infty}^{\infty} |g(x)| f_X(x) \, dx < \infty,
\]

then the expectation of \( Y \) exists and is \( E[Y] = \int_{-\infty}^{\infty} g(x) f_X(x) \, dx \).

(b) Suppose \( X \) is a discrete random variable with probability mass function \( p_X(x) \).
Suppose the support of \( X \) is denoted by \( S_X \). If \( \sum_{x \in S_X} |g(x)| p_X(x) < \infty \), then
the expectation of \( Y \) exists and it is given by \( E[Y] = \sum_{x \in S_X} g(x) p_X(x) \).
Example. Exercise 1.8.9.

Note. The text refers to the expectation “operator.” Crudely put, an operator is usually a mapping of a function to another function (differentiation is an example of an operator). A mapping of a function to a number is usually called a “functional” (definite integrals are examples of functionals). The next result show that expectation is a linear functional (…or “operator”). We use square brackets for expectations, which is common for operators.

Theorem 1.8.2. Let $g_1(X)$ and $g_2(X)$ be functions of a random variable $X$. Suppose the expectations of $g_1(X)$ and $g_2(X)$ exist. Then for any constants $k_1$ and $k_2$ the expectation of $k_1 g_1(X) + k_2 g_2(X)$ exists and it is given by

$$E[k_1 g_1(X) + k_2 g_2(X)] = k_1 E[g_1(X)] + k_2 E[g_2(X)].$$

Example. Exercise 1.8.7.