Section 1.8. Expectation of a Random Variables

Note. We now define the expectation (or "expected value") of both a discrete and continuous random variable.

Definition 1.8.1. Let X be a random variable. If X s a continuous random variable with probability density function f and if $\int_{-\infty}^{\infty} |x| f(x)\,dx < \infty$, then the *expectation* of X (or *expected value* or *mean*) is $E[X] = \int_{-\infty}^{\infty} x f(x)\,dx$. If X is a discrete random variable with probability mass function p and $\sum_x |x| p(x) < \infty$, then the *expectation* of X (or *expected value* or *mean*) is $E[X] = \sum_x x p(x)$. Sometimes (especially when using the term "mean") we denote $\mu = E(X)$.

Note. The hypothesis $\int_{-\infty}^{\infty} |x| f(x)\,dx < \infty$ is a hypothesis that function $x f(x)$ is "absolutely integrable." If a function is absolutely integrable then it is integrable (that is, if $\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$ then $\int_{-\infty}^{\infty} f(x)\,dx < \infty$). This follows from Corollary 6-15(b) of my Analysis 1 (MATH 4217/5217) notes on 6-2. Some Properties and Applications of the Riemann Integral (this holds for Lebesgue integrals; see Proposition 4.16. The Integral Comparison Test from my Real Analysis 1 [MATH 5201] on 4.4. The General Lebesgue Integral). So the hypothesis guarantees that the integral $\int_{-\infty}^{\infty} |x| f(x)\,dx$ exists as a finite value (recall that probability density function satisfies $|f(x)| = f(x) \geq 0$ for all $x \in \mathbb{R}$). The hypothesis $\sum_x |x| p(x) < \infty$ is a hypothesis of absolute summability of the series. Since the absolute convergence of a series implies the convergence of the series (see Theorem 10. The Absolute Convergence Test from my Calculus 2 [MATH 1920] notes on 10.6. Alternating Series, Absolute and Conditional Convergence) ... this also guarantees that the series can be rearranged (see Theorem 17. The Rearrangement Theorem for Absolutely Convergent Series in the same section of notes) so that the ambiguity of "$\sum_x$" is irrelevant (i.e., the terms can be summed in any order).

Example. If a fair 6-sided die is rolled and the random variable X is the outcome then the expected outcome is

$$E(X) = (1/6)(1) + (1/6)(2) + (1/6)(3) + (1/6)(4) + (1/6)(5) + (1/6)(6) = 7/2.$$

Note/Definition. A game of chance is a *fair game* is the expected gain (that is, the payoff minus the cost of playing) is 0.

Note. If we transform a random variable X to get $Y = g(X)$ then we can use the following theorem to compute $E[Y]$.

Theorem 1.8.1. Let X be a random variable and let $Y = g(X)$ for some function g.

(a) Suppose X is continuous with probability density function $f_X(x)$. If

$$\int_{-\infty}^{\infty} |g(x)| f_X(x)\, dx < \infty,$$

then the expectation of Y exists and is $E[Y] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$.

(b) Suppose X is a discrete random variable with probability mass function $p_X(x)$. Suppose the support of X is denoted by $\mathcal{S}_X$. If $\sum_{x \in \mathcal{S}_X} |g(x)| p_X(x) < \infty$, then the expectation of Y exists and it is given by $E[Y] = \sum_{x \in \mathcal{S}_X} g(x) p_X(x)$.

Example (Exercise 1.8.9). Let $f(x) = 2x$, $0 < x < 1$, zero elsewhere, be the pdf of X.

(a) Compute $E(1/X)$.

(b) Find the cdf and the pdf of $Y = 1/X$.

(c) Compute $E(Y)$ directly from the pdf of Y.

Note. The text refers to the expectation "operator." Crudely put, an operator is usually a mapping of a function to another function (differentiation is an example of an operator). A mapping of a function to a number is usually called a "functional" (definite integrals are examples of functionals). The next result show that expectation is a linear functional (...or "operator"). We use square brackets for expectations, which is common for operators.

Theorem 1.8.2. Let $g_1(X)$ and $g_2(X)$ be functions of a random variable X. Suppose the expectations of $g_1(X)$ and $g_2(X)$ exist. Then for any constants k_1 and k_2 the expectation of $k_1 g_1(X) + k_2 g_2(X)$ exists and it is given by

$$E[k_1 g_1(X) + k_2 g_2(X)] = k_1 E[g_1(X)] + k_2 E[g_2(X)].$$

Example (Exercise 1.8.7). Let X have the pdf $f(x) = 3x^2$, $0 < x < 1$, zero elsewhere. Consider a random rectangle where sides are X and $(1 - X)$. Determine the expected value of the area of the rectangle.