Section 2.5. The Correlation Coefficient

Note. We now introduce a parameter ρ of the joint distribution of (X, Y) which quantifies the dependence between X and Y (so that $\rho = 0$ when X and Y are independent). We assume the existence of all expectations under discussion.

Definition 2.5.1. Let (X, Y) have a joint distribution. Denote the means of X and Y respectively by μ_1 and μ_2 and their respective variances by σ_1^2 and σ_2^2. The *covariance* of (X, Y) is

$$\text{cov}(X, Y) = E[(X - \mu_1)(Y - \mu_2)].$$

Note 2.5.A. Since the expectation operator is linear by Theorem 2.1.1, then

$$\text{cov}(X, Y) = E[XY - \mu_2 X - \mu_1 Y + \mu_1\mu_2] = E[XY] - \mu_2 E[X] - \mu_1 E[Y] + \mu_1\mu_2$$

$$= E[XY] - \mu_1\mu_2 - \mu_1\mu_2 + \mu_1\mu_2 = E[XY] - \mu_1\mu_2.$$

Definition 2.5.2. If each of σ_1 and σ_2 is positive then the *correlation coefficient* between X and Y is

$$\rho = \frac{E[(X - \mu_1)(Y - \mu_2)]}{\sigma_1\sigma_2} = \frac{\text{cov}(X, Y)}{\sigma_1\sigma_2}.$$

Note 2.5.B. We can relate these parameters as

$$\begin{aligned} E[XY] &= \mu_1\mu_2 + \text{cov}(X, Y) \text{ by Note 2.5.A} \\ &= \mu_1\mu_2 + \rho\sigma_1\sigma_2 \text{ by Definition 2.5.2.} \end{aligned}$$

Example 2.5.2. Let X and Y have joint probability density function

$$f(x,y) = \begin{cases} x+y & \text{for } 0 < x < 1,\ 0 < 1 < y \\ 0 & \text{elsewhere.} \end{cases}$$

We have

$$\begin{aligned} \mu_1 = E[X] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)\,dx\,dy = \int_0^1\int_0^1 x(x+y)\,dx\,dy \\ &= \int_0^1 \left(\frac{1}{3}x^3 + \frac{1}{2}x^2y\right)\Big|_{x=0}^{x=1} dy = \int_0^1 \left(\frac{1}{3} + \frac{1}{2}y\right) dy \\ &= \left(\frac{1}{3}y + \frac{1}{4}y^2\right)\Big|_0^1 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}, \end{aligned}$$

similarly, $\mu_2 = E[Y] = 7/12$,

$$\begin{aligned} \sigma_1^2 &= E[X^2] - \mu_1^2 \text{ by Note 1.9.A} \\ &= \int_0^1\int_0^1 x^2(x+y)\,dx\,dy - \left(\frac{7}{12}\right)^2 = \int_0^1\int_0^1 (x^3 + x^2y)\,dx\,dy - \left(\frac{7}{12}\right)^2 \\ &= \int_0^1 \left(\frac{1}{4}x^4 + \frac{1}{3}x^3y\right)\Big|_{x=0}^{x=1} dx - \left(\frac{7}{12}\right)^2 = \int_0^1 \left(\frac{1}{4} + \frac{1}{3}y\right) dy - \left(\frac{7}{12}\right)^2 \\ &= \left(\frac{1}{4}y + \frac{1}{6}y^2\right)\Big|_{x=0}^{x=1} - \left(\frac{7}{12}\right)^2 = \frac{5}{12} - \frac{49}{144} = \frac{11}{144}, \end{aligned}$$

similarly $\sigma_2^2 = E[Y^2] - \mu_2^2 = 11/144$, and

$$\begin{aligned} \text{cov}(XY) &= E[XY] - \mu_1\mu_2 \text{ by Note 2.5.A} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)\,dx\,dy - \mu_1\mu_2 = \int_0^1\int_0^1 xy(x+y)\,dx\,dy - \mu_1\mu_2 \\ &= \int_0^1 \left(\frac{1}{3}x^3y + \frac{1}{2}x^2y^2\right)\Big|_{x=0}^{x=1} dy - \mu_1\mu_2 = \int_0^1 \left(\frac{1}{3}y + \frac{1}{2}y^2\right) dy - \mu_1\mu_2 \\ &= \left(\frac{1}{6}y^2 + \frac{1}{6}y^3\right)\Big|_0^1 - \mu_1\mu_2 = \frac{1}{3} - -\left(\frac{7}{12}\right)\left(\frac{7}{12}\right) = \frac{48-49}{144} = \frac{-1}{144}. \end{aligned}$$

So by Definition 2.5.2,

$$\rho = \frac{\text{cov}(XY)}{\sigma_1\sigma_2} = \frac{-1/144}{\sqrt{11/144}\sqrt{11/144}} = \frac{-1}{11}.$$

Theorem 2.5.1. For all jointly distributed random variables (X, Y) whose correlation coefficient ρ exists (so that $\sigma_1 > 0$ and $\sigma_2 > 0$ by the definition of ρ), we have $-1 \le \rho \le 1$.

Theorem 2.5.2. If X and Y are independent random variables then $\text{cov}(X, Y) = 0$ and hence $\rho = 0$.

Example 2.5.3. The converse of Theorem 2.5.2 does not in general hold. That is, we may have $\rho = 0$ where X and Y are dependent. Suppose X and Y have a joint probability mass distribution such that the four points $(-1, 0)$, $(0, -1)$, $(1, 0)$, and $(0, 1)$ have probability $1/4$ (and the probability is 0 for the other possible values of (x, y)). Then both X and Y have range $\{-1, 0, 1\}$ with respective marginal probabilities $1/4$, $1/2$, and $1/4$. So $\mu_1 = \mu_2 = 0$ and $E[XY] = (1/4)(-1)(0) + (1/4)(0)(-1) + (1/4)(1)(0) + (1/4)(0)(1) = 0$. So by Note 2.5.B $\text{cov}(X, Y) = E[XY] - \mu_1\mu_2 = 0$ (notice $\sigma_1^2 = \sigma_2^2 = 1/2 \neq 0$). However $P(X = 0, Y = 0) = 0$ while $P(X = 0)P(Y = 0) = (1/2)(1/2) = 1/4$. So $P(X = 0, Y = 0) \neq P(X = 0)P(Y = 0)$ and hence X and Y are dependent but $\rho = \text{cov}(X, Y)/(\sigma_1\sigma_2) = 0$.

Note. In Exercise 2.5.7 it is to be shown that if $\rho = 1$ then $Y = (\sigma_2/\sigma_1)X - (\sigma_2/\sigma_1)\mu_1 + \mu_2$ with probability 1 and if $\rho = -1$ then $Y = -(\sigma_2/\sigma_1)X + (\sigma_2/\sigma_1)\mu_1 + \mu_2$ with probability 1. So if the correlation coefficient is ± 1 then Y is a "linear function" (that is, a function of the form $mX + b$) of Y; the slope is positive if $\rho = 1$ and negative if $\rho = -1$. More generally, Hogg, McKean, and Craig comment "we can look upon ρ as a measure of the intensity of the concentration of the of the probability for X and Y about" a line (page 128). This is spelled out more formally in the next theorem.

Theorem 2.5.3. Suppose (X, Y) have a joint distribution with the variances of X and Y finite and positive. Denote the means and variances of X and Y by μ_1, μ_2 and σ_1^2, σ_2^2, respectively, and let ρ be the correlation coefficient between X and Y. If $E[Y \mid X]$ is linear in X then

$$E[Y \mid X]\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(X - \mu_1) \text{ and } E[\mathrm{Var}(Y \mid X)] = \sigma_2^2(1 - \rho^2).$$

Example 2.5.5. We now consider an example that illustrates how the correlation coefficient ρ reflects how the values of X and Y are concentrated along a line. Consider the joint probability density function

$$f(x,y) = \begin{cases} \frac{1}{4ah} & \text{for } -1 + bx < y < a + bx,\ -h < x < h \\ 0 & \text{elsewhere.} \end{cases}$$

The support of f is as given in Figure 2.5.1.

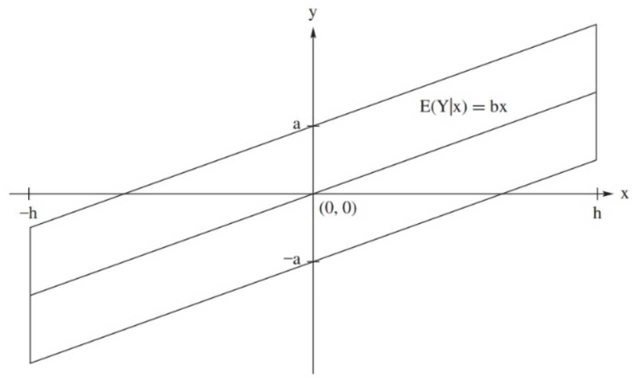

Figure 2.5.1 from page 130.

For the sake of illustration, we assume the slope satisfies $b \geq 0$. The marginal probability density function of X is

$$f_1(x) = \begin{cases} \int_{-a+bx}^{a+bx} \frac{1}{4ah}\,dy & \text{for } -h < x < h \\ 0 & \text{elsewhere} \end{cases} = \begin{cases} \frac{1}{2h} & \text{for } -h < x < h \\ 0 & \text{elsewhere} \end{cases}$$

so that f_1 is a uniform distribution (as is f). As shown in the proof of Theorem 2.5.3,

$$E[Y \mid X] = \frac{1}{f_1(x)} \int_{-\infty}^{\infty} y f(x,y)\,dy = \frac{1}{1/(2h)} \int_{-a+bx}^{a+bx} y \frac{1}{4ah}\,dy = \frac{1}{2a}\left(\frac{1}{2}y^2\right)\Bigg|_{y=-a+bx}^{y=a+bx}$$

$$= \frac{1}{4a}((a+bx)^2 - (-a+bx)^2) = \frac{1}{4a}(a^2 + 2abx + b^2x^2 - a^2 + 2abx - b^2x^2) = bx.$$

For $\text{var}(Y \mid x)$ we use the conditional mean of Y given x of $E[Y \mid x] = bx$ and have

$$\text{var}(Y \mid x) = \int_{-\infty}^{\infty} (y - bx)^2 f_{2|1}(y \mid x), dy = \int_{y=-a+bx}^{y=a+bx} (y-bx)^2 \frac{f(x,y)}{f_1(x)}\,dy$$

$$= \int_{y=-a+bx}^{y=a+bx} (y-bx)^2 \frac{1/(4ah)}{1/(2h)}\,dy = \frac{1}{2a}\frac{1}{3}(y-bx)^3\Bigg|_{y=-a+bx}^{y=a+bx} = \frac{1}{6a}(a^3 - (-a)^3) = \frac{a^2}{3}.$$

Since $E[Y \mid x] = bx$ then $E[Y \mid X]$ is a linear function of X and so Theorem 2.5.3 holds from which we see that $E[Y \mid X] = \mu_1 + \rho\frac{\sigma_2}{\sigma_1}(X - \mu_1) = bX$ and hence $\mu_1 =$

$\mu_2 = 0$ and $b = \rho\sigma_2/\sigma_1$. Also by Theorem 2.5.3, $E[\text{var}(Y \mid X)] = \sigma_2^2(1-\rho^2) = a^2/3$. Next,

$$\sigma_1^2 = \int_{-\infty}^{\infty} (x-\mu_1)^2 f_1(x)\,dx = \int_{-h}^{h} x^2 \frac{1}{2h}\,dx = \frac{1}{2h}\left(\frac{1}{3}x^3\right)\Bigg|_{-h}^{h} = \frac{1}{6h}(h^3-(-h)^3) = \frac{h^2}{3}.$$

We solve the three equations

$$(\mathbf{1})\ \ b = \rho\frac{\sigma_2}{\sigma_1}, \quad (\mathbf{2})\ \ \frac{a^2}{3} = \sigma_2^2(1-\rho^2), \quad (\mathbf{3})\ \ \sigma_1^2 = \frac{h^2}{2}$$

for ρ in terms of a, b, h then we get from (1) that $\sigma_2 = b\sigma_1/\rho$ and so from (2) that

$$\frac{a^2}{3} = \left(\frac{b\sigma_1}{\rho}\right)^2 (1-\rho^2) = b^2\sigma_1^2\left(\frac{1}{\rho^2} - 1\right) = \frac{b^2h^2}{3}\left(\frac{1}{\rho^2} - 1\right)$$

from (3). So $\dfrac{a^2}{b^2h^2} = \dfrac{1}{\rho^2} - 1$ or $\dfrac{a^2}{b^2h^2} + 1 = \dfrac{a^2 +^2 h^2}{b^2h^2} = \dfrac{1}{\rho^2}$ or $\rho^2 = \dfrac{b^2h^2}{a^2+b^2h^2}$ and $\rho = \dfrac{bh}{\sqrt{a^2+b^2h^2}}$ (we have $b \geq 0$ and $h > 0$ and, since $b = \rho\sigma_2/\sigma_1$, $\rho \geq 0$). From the equation $\rho = \dfrac{bh}{\sqrt{a^2+b^2h^2}}$ and Figure 2.5.1 we have

1. As a gets smaller (respectively, larger), the straight-line effect is more (respectively, less) intense and ρ is closer to 1 (respectively, closer to 0).

2. As h gets larger (respectively, smaller), the straight-line effect is more (respectively, less) intense and ρ is closer to 1 (respectively, closer to 0).

3. As b gets larger (respectively, smaller), the straight-line effect is more (respectively, less) intense and ρ is closer to 1 (respectively, closer to 0).

Note 2.5.C. In Section 2.1 we saw that the moment generating function

$$M_{X,Y}(t_1, t_2) = E[e^{\mathbf{t}'\mathbf{X}}] = E[e^{t_1X+t_2Y}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1x+t_2y} f(x,y)\,dy\,dx$$

so by Proposition IV.2.1 of my Complex Analysis notes on IV.2. Power Series Representations of Analytic Functions (which requires the integrand to have first partial derivatives)

$$\frac{\partial^{k+m} M(t_1,t_2)}{\partial t_1^k\,\partial t_2^m} = \frac{\partial^{k+m}}{\partial t_1^k\,\partial t_2^m}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1x+t_2y} f(x,y)\,dy\,dx\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial^{k+m}}{\partial t_1^k\,\partial t_2^m}\left[e^{t_1x+t_2y} f(x,y)\right]\,dy\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^k y^m e^{t_1x+t_2y} f(x,y)\,dy\,dx.$$

With $t_0 = t_2 = 0$ we have

$$\left.\frac{\partial^{k+m} M(t_1,t_2)}{\partial t_1^k\,\partial t_2^m}\right|_{t_1=t_2=0} = \int_{-\infty}^{\infty} x^k y^m f(x,y)\,dy\,dx = E[X^kY^m].$$

This allows us to calculate several parameters using the moment generating function:

$$\begin{aligned}
\mu_1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)\,dy\,dx = \frac{\partial M(0,0)}{\partial t_1}\\
\mu_2 &= \int_{-\infty}^{\infty} y f(x,y)\,dx\,dy = \frac{\partial M(0,0)}{\partial t_2}\\
\sigma_1^2 &= E[X^2] - \mu_1^2 \text{ by Note 1.9.A}\\
&= \int_{-\infty}^{\infty}\int_{\infty}^{\infty} x^2 f(x,y)\,dx\,dy - \mu_1^2 = \frac{\partial M(0,0)}{\partial t_1^2} = \mu_1^2\\
\sigma_2^2 &= E[Y^2] - \mu_2^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2 f(x,y)\,dx\,dy - \mu_2^2 = \frac{\partial M(0,0)}{\partial t_2^2} - \mu_2^2\\
\text{cov}(X,Y) &= E[(X-\mu_1)(T-\mu_2)] = E[XY] - \mu_1\mu_2 \text{ by Note 2.5.A}\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f(x,y)\,dx\,dy - \mu_1\mu_2 = \frac{\partial^2 M(0,0)}{\partial t_1\,\partial t_2} - \mu_1\mu_2.
\end{aligned}$$

Since the correlation coefficient is $\rho = \dfrac{\text{cov}(X,Y)}{\sigma_1\sigma_2}$ then we can also calculate ρ using the joint moment generating function.

Example 2.5.6. Consider the joint probability density function

$$f(x,y) = \begin{cases} e^{-y} & \text{for } 0 < x < y < \infty \\ 0 & \text{elsewhere,} \end{cases}$$

which was explored in the class notes in Example 2.1.10 and it was shown that

$$M(t_1,t_2) = \frac{1}{(1-t_2)(t-t_1-t_2)} = \frac{1}{1-t_1-2t_2+t_1t_2+t_2^2}$$

for $t_1 + 2t_2 - 1 < 0$ and $t_2 < 1$. So we have

$$\begin{aligned}
\mu_1 &= \frac{\partial M(0,0)}{\partial t_1} = \left.\frac{1}{(1-t_2)(1-t_1-t_2)^2}\right|_{t_1=t_2=0} = 1 \\
\mu_2 &= \frac{\partial M(0,0)}{\partial t_2} = \left.\frac{-(-2+t_1+2t_2)}{(1-t_1-2t_2+t_1t_2+t_2^2)^2}\right|_{t_1=t_2=0} = 2 \\
\sigma_1^2 &= \frac{\partial^2 M(0,0)}{\partial t_1^2} - \mu_1^2 = \left.\frac{2}{(1-t_2)(1-t_1-t_2)^3}\right|_{t_1=t_2=0} - \mu_1^2 = 2 - 1^2 = 1 \\
\sigma_2^2 &= \frac{\partial^2 M(0,0)}{\partial t_2^2} = \big\{[-2](1-t_1-2t_2+t_1t_2+t_2^2)^2 \\
&\quad -(2-t_1-2t_2)[2(1-t_1-2t_2+t_1t_2+t_2^2)^2[-2+t_1+2t_2]\big\} \\
&\quad \div(1-t_1-2t_2+t_1t_2+t_2^2)^2\big|_{t_1=t_2=0} - \mu_2^2 = \frac{-2+8}{1} - 2^2 = 2, \\
\operatorname{cov}(X,Y) &= \frac{\partial^2 M(0,0)}{\partial t_1 \partial t_2} - \mu_1\mu_2 = \frac{\partial}{\partial t_1}\left.\left[\frac{2-t_1-2t_2}{(1-t_1-2t_2+t_1t_2+t_2^2)^2}\right]\right|_{t_1=t_2=0} \\
&= \big\{[-1](1-t_1-2t_2+t_1t_2+t_2^2)^2 \\
&\quad -(2-t_1-2t_2)[2(1-t_1-2t_2+t_1t_2+t_2^2)[-1+t_2]]\big\} \\
&\quad \div(1-t_1-2t_2+t_1t_2+t_2^2)^4\big|_{t_1=t_2=0} \\
&= \frac{(-1)-(2)(-2)}{1} - (1)(2) = 3 - 2 = 1.
\end{aligned}$$

(The computation of these values is Exercise 2.5.5.) So

$$\rho = \frac{\operatorname{cov}(X,Y)}{\sigma_1\sigma_2} = \frac{1}{1\sqrt{2}} = \frac{1}{\sqrt{2}}.$$