Section 2.8. Linear Combinations of Random Variables

Note. We considered linear combinations of random variables in Section 2.6, "Extension to Several Variables," in connection with moment generating functions (see Theorem 2.6.1 and Corollary 2.6.1). In this section we consider expectations of random variables, variances, and covariances of linear combinations of random variables. With $X_1, X_2, \ldots, X_n$ as the random variables, we define random variable $T = \sum_{i=1}^n a_i X_i$ where each a_i is some constant.

Theorem 2.8.1. Let $X_1, X_2, \ldots, X_n$ be random variables and define $T = \sum_{i=1}^n a_i X_i$. Suppose $E[X_i] = \mu_i$ for $i = 1, 2, \ldots, n$. Then $E[T] = \sum_{i=1}^n a_i \mu_i$.

Note. To find the variance of T, we first consider covariances.

Theorem 2.8.2. Let $X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_m$ be random variables and define $T = \sum_{i=1}^n a_i X_i$ and $W = \sum_{j=1}^m b_j Y_j$. If $E[X_i^2] < \infty$ and $E[Y_j^2] < \infty$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, then

$$\mathrm{Cov}(T, W) = \sum_{i=1}^n \sum_{j=1}^m a_i b_j \,\mathrm{Cov}(X_i, Y_j).$$

Corollary 2.8.1. Let $X_1, X_2, \ldots, X_n$ be random variables and define $T = \sum_{i=1}^n a_i X_i$. Provided $E[X_i^2] < \infty$ for $i = 1, 2, \ldots, n$, then

$$\mathrm{Var}(T) = \sum_{i=1}^n a_i^2 \,\mathrm{Var}(X_i) + s \sum_{i<j} a_i a_j \,\mathrm{Cov}(X_i, X_j).$$

Corollary 2.8.2. Let $X_1, X_2, \ldots, X_n$ be independent random variables and define $T = \sum_{i=1}^n a_i X_i$. With $\mathrm{Var}(X_i) = \sigma_i^2$ for $i = 1, 2, \ldots, n$ we have $\mathrm{Var}(T) = \sum_{i=1}^n a_i^2 \sigma_i^2$.

Note. We now give definitions which are our first step from mathematical probability to mathematical statistics.

Definition 2.8.1. If the random variables $X_1, X_2, \ldots, X_n$ are independent and identically distributed; that is, each X_i has the same distribution, then these random variables constitute a *random sample* of size n from that common distribution. We abbreviate independent and identically distributed as "iid."

Definition. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from some distribution with mean μ and variance σ^2. The *sample mean* is $\overline{X} = \dfrac{\sum_{i=1}^n X_i}{n}$. The *sample variance* is $S^2 = \dfrac{\sum_{i=1}^n (X_i - \overline{X})^2}{n-1}$.

Theorem 2.8.A. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with common mean μ and variance σ^2. We have

$$E[\overline{X}] = \mu,\ \mathrm{Var}(\overline{X}) = \frac{\sigma^2}{n},\ S^2 = \frac{\sum_{i=1}^n X_i^2 - n\overline{X}^2}{n-1},\ \text{and } E[S^2] = \sigma^2.$$

Note. Since $E[\overline{X}] = \mu$ and $E[S^2] = \sigma^2$, then $\overline{X}$ is an *unbiased* estimate of μ, and S^2 is an unbiased estimate of σ^2.

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