Introduction to Modern Algebra

Supplement. The Cayley-Dickson Construction and Nonassociative Algebras—Proofs of Theorems

Theorem CD.3

Theorem CD.3. If algebra A is nicely normed and alternative, then A is a normed division algebra.

Proof. Recall that a normed division algebra is an algebra that is also a normed vector space with $\|ab\| = \|a\|\|b\|$ for all $a, b \in A$. If A is nicely normed, then $aa^* = a^*a > 0$ and the norm is given by $\|a\| = \sqrt{aa^*}$. Let $a, b \in A$. Then the subalgebra generated by $\mathrm{Im}(a)$ and $\mathrm{Im}(b)$ is associative (because A is alternative; see the definition of "alternative") and includes a, b, a^*, b^*. We now have

$$\begin{aligned} \|ab\|^2 &= (ab)(ab)^* \text{ by the definition of the norm} \\ &= (ab)(b^*a^*) \text{ by the definition of conjugation} \\ &= ((ab)b^*)a = (a(bb^*))a^* \text{ by associativity} \\ &= (a\|b\|^2)a^* \text{ by the definition of the norm} \end{aligned}$$

Theorem CD.3 (continued)

Theorem CD.3. If $*$-algebra A is nicely normed and alternative, then A is a normed division algebra.

Proof (continued). ...

$$\begin{aligned} \|ab\|^2 &= (a\|b\|^2)a^* \text{ by the definition of the norm} \\ &= a(\|b\|^2a^*) \text{ by associativity} \\ &= aa^*\|b\|^2 \text{ since } \|b\|^2 \text{ is real and so commutes with } a^* \\ &= \|a\|^2\|b\|^2 \text{ by the definition of the norm.} \end{aligned}$$

Therefore $\|ab\| = \|a\|\|b\|$ and so A is a normed division algebra, as claimed. □

Proposition 1

Proposition 1. Starting with any $*$-algebra A, the new $*$-algebra A' that results from the Cayley-Dickson Construction is not real.

Proof. Let $a, b \in A$ and consider $(a, b) \in A'$. If (a, b) is real, then we must have $(a, b) = (a, b)^* = (a^*, -b)$. But this implies that $a = a^*$ (so that a must be real) and $b = -b$ (so that b must be $0 \in A$). We take A to contain more than just 0, so that we do not have $(a, b) = (a, b)^*$ for all $a, b \in A$. That is, A' is not real, as claimed. □

Proposition 2

Proposition 2. $*$-algebra A is real (and thus commutative) if and only if Cayley-Dickson algebra A' constructed from A is commutative.

Proof. First, suppose that $*$-algebra A is real (that is, $a = a^*$ for all $a \in A$). Then for $(a, b), (c, d) \in A'$, we have the product:

$$\begin{aligned} (a,b)(c,d) &= (ac - db^*, a^*d + cb) \text{ by definition of multiplication in } A' \\ &= (ca - bd^*, c^*b + ad) \text{ since } A \text{ is real and commutative} \\ &= (c,d)(a,b). \end{aligned}$$

Since (a, b) and (c, d) are arbitrary elements of A', then A' is commutative as claimed.

Next, suppose A' is commutative. Then for $(a, b), (c, d) \in A'$, we must have $(a, b)(c, d) = (c, d)(a, b)$; that is, $(ac - db^*, a^*d + cb) = (ca - bd^*, c^*b + ad)$. In particular, with $b = d = 0$ we have for all $a, c \in A$ that $(ac, 0) = (ca, 0)$. That is, $ac = ca$ for all $a, c \in A$ and so A is commutative, as claimed.

Proposition 2 (continued)

Proposition 2. $*$-algebra A is real (and thus commutative) if and only if Cayley-Dickson algebra A' constructed from A is commutative.

Proof (continued). With $a = c = 0$ we have for all $b, d \in A$ that $(-db^*, 0) = (-bd^*, 0)$. That is, $bd^* = db^*$ for all $b, d \in A$. Now by the definition of conjugation,

$$(db^*)^* = b^{**}d^* = bd^* = db^*$$

and so db^* is real for all $b, d \in A$. With $d = 1$, we have b^* is real for all $b \in A$; that is, $b^* = b$ is real for all $b \in A$ and so A is real, as claimed. □

Proposition 3

Proposition 3. $*$-algebra A is commutative and associative if and only if Cayley-Dickson algebra A' constructed from A is associative.

Proof. Suppose A is commutative and associative. Let $(a, b), (c, d), (e, f) \in A'$. Then

$$\big((a,b)(c,d)\big)(e,f) = \big(ac - cb^*, a^*d + cb\big)(e,f)$$

$$\begin{aligned} &= \big((ac - db^*)e - f(a^*d + cb)^*, (ac - db^*)^*f + e(a^*d + cd)\big) \\ &= \big((ac)e - (db^*)e - f(d^*a + b^*c^*), (c^*a^* - bd^*)f + e(a^*d) + e(cb)\big) \\ &= \big(((ac)e - (db^*)e - f(d^*c^*), (c^*a^*)f - (bd^*)f_e(a^*d) + e(cb)\big) \\ &= \big((ac)e - f(d^*a) - f(b^*c^*) - (db^*)e, \\ &\qquad (c^*a^*)f + e(a^*d) + e(cb) - (bd^*)f\big) \\ &= \big(a(ce) - (fd^*)a - (fb^*)c^* - (db^*)e, (c^*a^*)f + (ea^*)d \\ &\qquad +(ec)b - b(d^*f)\big) \text{ by associativity in } A \end{aligned}$$

Proposition 3 (continued 1)

Proposition 3. $*$-algebra A is commutative and associative if and only if Cayley-Dickson algebra A' constructed from A is associative.

Proof (continued). $\ldots \big((a,b)(c,d)\big)(e,f) =$

$$\begin{aligned} &= \big(a(ce) - a(fd^*) - c^*(fb^*) - e(db^*), (a^*c^*)f + (a^*e)d + (ce)b \\ &\qquad -(fd^*)b\big) \text{ by commutativity in } A \text{ (applied twice in the last product)} \\ &= \big(a(ce) - a(fd^*) - (c^*f)c^* - (ed)b^*, a^*(c^*f) + a^*(ed) + (ce)b \\ &\qquad -(fd^*)b\big) \text{ by associativity in } A \\ &= \big(a(ce - fd^*) - (c^*f + ed)b^*, a^*(c^*f + ed) + (ce - fd^*)b\big) \\ &= (a,b)(ce - fd^*, c^*f + ed) = (a,b)\big((c,d)(e,f)\big). \end{aligned}$$

That is, $\big((a,b)(c,d)\big)(e,f) = (a,b)\big((c,d)(e,f)\big)$ and A is associative, as claimed.

Proposition 3 (continued 2)

Proposition 3. $*$-algebra A is commutative and associative if and only if Cayley-Dickson algebra A' constructed from A is associative.

Proof (continued). Suppose A' is associative. Then as computed above,

$$\big((a,c)e - f(d^*a) - f(b^*c^*) - (db^*)e, (c^*a^*)f + e(a^*d) + e(cb\_(bd^*)f\big)$$

$$= \big(a(ce) - a(fd^*) - (c^*f)b^* - (ed)b^*, a^*(c^*f) + a^*(ed) + (ce)b - (fd^*)b\big)$$

for all $a,b,c,d,e,f \in A$. With $b = d = f = 0$ this implies $\big((ac)e, 0\big) = \big(a(ce), 0\big)$ and hence $(ac)e = a(ce)$ for all $a,c,e \in A$; that is, A is associative, as claimed. With $b = c = e = 0$ an $dd = 1$ we have $(-fa, 0) = (-af, 0)$ and hence $-fa = -af$ or $af = fa$ for all $a,f \in A$; that is, A is commutative, as claimed. □

Proposition 4

Proposition 4. $*$-algebra A is associative and nicely normed if and only if Cayley-Dickson algebra A' constructed from A is alternative and nicely normed.

Proof. Suppose A is associative and nicely normed. Let $(a,b),(c,d) \in A'$. Then

$$\big((a,b)(a,b)\big)(c,d) = (aa - bb^*, a^*b + ab)(c,d)$$

$$\begin{aligned}
&= \big((aa - bb^*)c - d(a^*b + ab)^*, (aa - bb^*)^*d + c(a^*b + ab)\big)\\
&= \big((aa)c - (bb^*)c - d(b^*a + b^*a^*), (a^*a^* - bb^*)d + c(a^*b + ab)\big)\\
&= \big((aa)c - (bb^*)c - db^*(a + a^*), (a^*a^*)d - (bb^*)d + c(a^* + a)b\big)\\
&= \big((aa)c - c(bb^*) - (a + a^*)db^*, (a^*a^*)d - d(bb^*) + (a^* + a)cb\big)\\
&\quad \text{since } bb^* \text{ and } a + a^* \text{ are real by the definition of nicely normed}\\
&= \big((aa)c - a(db^*) - a^*(db^*) - c(bb^*), (a^*a^*)d + a^*(cb) + a(cb)\\
&\quad -d(b^*b)\big) \text{ since } bb^* = b^*b \text{ because } A \text{ is nicely normed}
\end{aligned}$$

Proposition 4 (continued 1)

Proposition 4. $*$-algebra A is associative and nicely normed if and only if Cayley-Dickson algebra A' constructed from A is alternative and nicely normed.

Proof (continued). ...

$$\big((a,b)(a,b)\big)(c,d) = (aa - bb^*, a^*b + ab)(c,d)$$

$$\begin{aligned}
&= \big((aa)c - a(db^*) - a^*(db^*) - c(bb^*), (a^*a^*)d + a^*(cb) + a(cb)\\
&\quad -d(b^*b)\big) \text{ since } bb^* = b^*b \text{ because } A \text{ is nicely normed}\\
&= \big(a(ac) - a(db^*) - (a^*d)b^* - (cb)b^*, a^*(a^*d) + a^*(cb) + (ac)b\\
&\quad -(db^*)b\big) \text{ since } A \text{ is associative}\\
&= \big(a(ac - db^*) - (a^*d + cb)b^*, a^*(a^*d + cb) + (ac - db^*)b\big)\\
&= (a,b)(ac - db^*, a^*d + cb) = (a,b)\big((a,b)(c,d)\big).
\end{aligned}$$

That is, $(aa)b = a(ab)$ for all $a, b \in A'$.

Proposition 4 (continued 2)

Proposition 4. $*$-algebra A is associative and nicely normed if and only if Cayley-Dickson algebra A' constructed from A is alternative and nicely normed.

Proof (continued). Next, let $(a,b),(c,d) \in A'$. Then

$$\big((c,d)(a,b)\big)(a,b) = (ca - bd^*, c^*b + ad)(a,b)$$

$$\begin{aligned}
&= \big((ca - bd^*)a - b(c^*b + ad)^*, (ca - bd^*)^*b + a(c^*b + ad)\big)\\
&= \big((ca - bd^*)a - b(b^*c + d^*a^*), (a^*c^* - db^*)b + a(c^*b + ad)\big)\\
&= \big((ca)a - (bd^*)a - b(b^*c) - b(d^*a^*), (a^*c^*)b - (db^*)b\\
&\quad +a(c^*b) + a(ad)\big)\\
&= \big(c(aa) - (bb^*)c - (bd^*)a - (bd^*)a^*, a^*(c^*b) + a(c^*b)\\
&\quad +(aa)d - d(b^*b)\big) \text{ since } A \text{ is associative}\\
&= \big(c(aa) - (bb^*)c - (bd^*)(a + a^*), (a^* + a)(c^*b) + (aa)d - d(bb^*)\big)\\
&\quad \text{since } b^*b = bb^* \text{ because } A \text{ is nicely normed}
\end{aligned}$$

Proposition 4 (continued 3)

Proof (continued). ...

$$\big((c,d)(a,b)\big)(a,b) = (ca - bd^*, c^*b + ad)(a,b)$$
$$= \big(c(aa) - (bb^*)c - (bd^*)(a+a^*), (a^*+a)(c^*b) + (aa)d - d(bb^*)\big)$$
$$= \big(c(aa) - (bb^*)c - (bd^*)(a+a^*), ((a^*+a)c^*)b + (aa)d - d(bb^*)\big)$$
by the associativity of A
$$= \big(c(aa) - (bb^*)c - (a+a^*)(bd^*), (c^*(a^*+a))b + (aa)d - d(bb^*)\big)$$
since $a + a^* \in \mathbb{R}$ and so commutes with elements of A
$$= \big(c(aa) - (bb^*)c - a(bd^*) - a^*(bd^*), (c^*a^*)b + (c^*a)b + (aa)d - d(bb^*)\big)$$
$$= \big(c(aa) - c(bb^*) - (a^*b)d^* - (ab)d^*, c^*(a^*b) + c^*(ab) + (aa)d - d(bb^*)\big)$$
since $bb^* = b^*b$ because A is nicely normed
$$= \big(c(aa - bb^*) - (a^*b + ab)d^*, c^*(a^*b + ab) + (aa - bb^*)d\big)$$
$$= (c,d)(aa - bb^*, a^*b + ab) = (c,d)\big((a,b)(a,b)\big).$$

Proposition 4 (continued 4)

Proposition 4. $*$-algebra A is associative and nicely normed if and only if Cayley-Dickson algebra A' constructed from A is alternative and nicely normed.

Proof (continued). That is, $(ba)a = b(aa)$ for all $a, b \in A'$. Therefore, by Schafer's Theorem 3.1, A' is alternative, as claimed.

We now show that A' is nicely normed. Since A is nicely normed then, by definition, $a + a^* \in \mathbb{R}$ and $aa^* = a^*a > 0$ for all nonzero $a \in A$. Let $(a,b) \in A'$. Then

$$(a,b) + (a,b)^* = (a,b) + (a^*, -b) \text{ be equation (3)}$$
$$= (a + a^*, b - b) = (a + a^*, 0)$$

and $(a + a^*, 0)^* = (a^* + a^{**}, 0) = (a + a^*, 0)$ so that $(a,b) + (a,b)^*$ is real for all $(a,b) \in A'$.

Proposition 4 (continued 5)

Proposition 4. $*$-algebra A is associative and nicely normed if and only if Cayley-Dickson algebra A' constructed from A is alternative and nicely normed.

Proof (continued). Also, for nonzero $(a,b) \in A'$, we have

$$(a,b)(a,b)^* = (a,b)(a^*, -b) = (aa^* - (-b)b^*, a^* - b) + a^*b)$$
$$= (aa^* + bb^*, -a^*b + a^*b) = (\|a\|^2 + \|b\|^2, 0)$$
$$= (a^*a + bb^*, ab - ab) = (a^*a - b(-b^*), ab + a(-b))$$
$$= (a^*, -b)(a,b) = (a,b)^*(a,b) > 0.$$

So, be definition, A' is nicely normed as claimed.

Finally, suppose A' is alternative and nicely normed. Then for all $a, b \in A$ we have $(a,b) + (a,b)^* = (a + a^*, 0)$ is real. That is,

$$(a + a^*, 0)^* = ((a + a^*)^*, 0) = (a^* + a^{**}, 0) = (a + a^*, 0).$$

Hence, $(a + a^*)^* = a + a^*$ (i.e., a is real) for all $a \in A$.

Proposition 4 (continued 6)

Proof (continued). Also, for nonzero $a \in A$ and for all $b \in A$ we have (as shown above)

$$(a,b)(a,b)^* = (a,b)^*(a,b) = (\|a\|^2 + \|b\|^2, 0) > 0.$$

In particular, with $b = 0$ we have that $aa^* = a^*a = \|a\|^2 > 0$. That is, A is nicely normed.

Since A' is alternative then, by Schafer's Theorem 3.1, we have for all $a, b, c, d \in A$ that $\big((a,b)(a,b)\big)(c,d) = (a,b)\big((a,b)(c,d)\big)$. As computed at the beginning of the proof (under the assumption that A is nicely normed), we have $\big((a,b)(a,b)\big)(c,d) = \big((aa)c - a(db^*) - a^*(db^*) - c(bb^*), (a^*a^*)d + a^*(cb) + a(cb) - d(b^*b)\big)$. By direct computation (also at the beginning of the proof) we have $(a,b)\big((a,b)(c,d)\big) = \big(a(ac) - a(db^*) - a^*d)b^* - (cb)b^*, a^*(a^*d) + a^*(cb) + (ac)b - (db^*)b\big)$.

Proposition 4 (continued 7)

Proposition 4. $*$-algebra A is associative and nicely normed if and only if Cayley-Dickson algebra A' constructed from A is alternative and nicely normed.

Proof (continued). So we must have

$$(a^*a^*)d+a^*(cb)+a(cb)-d(b^*b) = a^*(a^*d)+a^*(cb)+(ac)b-(db^*)b \quad (*)$$

Since A' is alternative then so is A (since it is isomorphic to a subalgebra of A'), and so $(a^*a^*)d = a^*(a^*d)$ and $d(b^*b) = (db^*)b$. So from $(*)$ we have $a(cb) = (ac)b$ for all $a, b, c \in A$, and hence A is associative as claimed. □

Proposition 5

Proposition 5. $*$-algebra A is nicely normed if and only if Cayley-Dickson algebra A' constructed from A is nicely normed.

Proof. In Proposition 4, it is shown that if A' is nicely normed then A is nicely normed.

Suppose A is nicely normed. Then for all nonzero $a \in A$ we have $a + a^* \in \mathbb{R}$ and $aa^* = a^*a > 0$. Let $(a, b) \in A'$. Then $(a, b) + (a, b)^* = (a, b) + (a^*, -b) = (a + a^*, 0)$ and since $(a + a^*, 0)^* = ((a + a^*)^*, -0) = (a + a^*, 0)$ then $(a, b) + (a, b)^*$ is real. Also, for nonzero $(a, b) \in A'$ we have $(a, b)(a, b)^* = (a, b)(a^*, -b) = (aa^* - (-b)b^*, a^*(-b) + a^*b) = (aa^* + bb^*, 0) = (\|a\|^2 + \|b\|^2, 0) > 0$ and $(a, b)^*(a, b) = (a^*, -b)(a, b) = (a^*a - b(-b^*), a^{**}b + a(-b)) = (aa^* + bb^*, ab - ab) = (\|a\|^2 + \|b\|^2, 0) = (a, b)(a, b)^*$. Therefore A' is nicely normed, as claimed. □