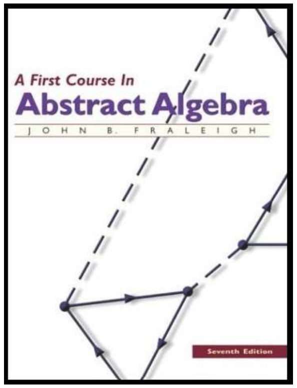
Introduction to Modern Algebra

Part IX. Factorization

VII.45. Unique Factorization Domains



Lemma 45.9.

Lemma 45.9. Let R be a commutative ring and let $N_1 \subseteq N_2 \subseteq ...$ be an ascending chain of ideals N_i in R. Then $N = sup_i\, N_i$ is an ideal of R.

Proof Let $a, b \in N$. Then there are ideals N_i and N_j in the chain with $a \in N_i$ and $b \in N_j$. WLOG, $N_i \subseteq N_j$ and $a, b \in N_j$. Every ideal is an additive subgroup, so $a \pm b \in N_j$. By the definition of ideal, $ab \in N_j$. So $a \pm b, ab \in N$.
Since $0 \in N_i$ for all i, it follows that for all $b \in N$, we have $-b \in N$ and $0 \in N$. By Exercise 18.48, N is a subring of R. For $a \in N$ and $r \in R$, we have $a \in N_i$ for some i and since N_i is an ideal, then $da = ad \in N_i$. So $ad \in \cup_i N_i$ and $da \in N$. So N is an ideal of R. □

Lemma 45.10. The Ascending Chain Condition for a PID

Lemma 45.10. The Ascending Chain Condition for a PID. Let D be a PID. If $N_1 \subseteq N_2 \subseteq ...$ is an ascending chain of ideals, then there exists a positive integer r such that $N_r = N_s$ for all $s \geq r$. Equivalently, every strictly ascending chain of ideals in a PID is of finite length. Under such conditions it is said that the *ascending chain condition* holds for ideals in a PID.

Proof. By Lemma 45.9, we have that $N = \cup_i N_i$ is an ideal of D. Since D is a PID then N is a principal ideal and so $N = \langle c \rangle$ for some $c \in D$. Since $N = \cup_i N_i$, then $c \in N_r$ for some $r \in \mathbb{N}$. For $s \geq r$ we have $\langle c \rangle \subseteq N_r \subseteq N_s \subseteq N = \langle c \rangle$. So $N_r = N_s$ for all $s \geq r$. □

Theorem 45.11.

Lemma 45.11. Let D be a PID. Every element that is neither 0 nor a unit of D is a product of irreducibles.

Proof. Let $a \in D$ where 'a' is neither 0 nor a unit. [We first show that 'a' has at least one irreducible factor.]
If 'a' itself is irreducible then we are done. If 'a' is not irreducible, then $a = a_1 b_1$ where neither a_1 or b_1 is a unit. Now $\langle a \rangle \subset \langle a_1 \rangle$ by Note 1 Part (1) (if $\langle a \rangle = \langle a_1 \rangle$ then by Note 1 Part (2) 'a' and a_1 would be associates, contradicting the fact that neither a_1 nor b_1 is a unit). If a_1 is irreducible then a_1 is an irreducible factor of 'a'. If not, write $a_1 = a_2 b_2$ where neither a_2 nor b_2 is a unit. As above, we have $\langle a_1 \rangle \subset \langle a_2 \rangle$. Continue this process to form a strictly ascending chain $\langle a \rangle \subset \langle a_1 \rangle \subset \langle a_2 \rangle \subset ....$
By Lemma 45.10, this chain terminates with some $\langle a_r \rangle$ and this a_r must be irreducible (or else we would contruct $\langle a_{r+1} \rangle$ with $\langle a_r \rangle \subset \langle a_{r+1} \rangle$). We now have $a = b_1 b_2 ... b_r a_r$ and so a_r is an irreducible factor of 'a'.

Theorem 45.11. (Continued)

Lemma 45.11. Let D be a PID. Every element that is neither 0 nor a unit of D is a product of irreducibles.

Proof. (Continued) Now that we know 'a' has an irreducible factor, we show that it can be written as a product of irreducible factors.
By above, we have that 'a' (neither 0 nor a unit in D) is irreducible or of the form $a = p_1 c_1$ for p_1 an irreducible and c_1 not a unit. If c_1 is not a unit (and of course it's not 0) then by the argument of the first paragraph we have $\langle a \rangle \subset \langle c_1 \rangle$ and if c_1 is not irreducible then $c_1 = p_2 c_2$ for irreducible p_2 with c_2 not a unit. Continuing we again get a strictly ascending chain of ideals $\langle a \rangle \subset \langle c_1 \rangle \subset \langle c_2 \rangle \subset \dots$.
By Lemma 45.10, this chain terminates with some $c_r = q_r$ that is irreducible (as argued in the first paragraph). Then $a = p - 1 p_2 \dots p_r q_r$ is a product of irreducibles. □

Lemma 45.12.

Lemma 45.12. An ideal $\langle p \rangle$ in a PID is maximal if and only if p is irreducible.

Proof. Let $\langle p \rangle$ be a maximal ideal of D, a PID. Suppose $p = ab$ in D. Then by Note 1 Part(1) $\langle p \rangle \subseteq \langle a \rangle$. If $\langle p \rangle = \langle a \rangle$ then by Note 1 Part(2) 'a' and p are associates and b is a unit. If $\langle p \rangle \neq \langle a \rangle$ then since $\langle p \rangle$ is maximal it must be that $\langle a \rangle = D$. From the definition of "ideal in D" we have $D = \langle 1 \rangle$, so in this case $\langle a \rangle = \langle 1 \rangle$ and by Note 1 Part(2), 'a' and 1 are associates and hence 'a' is a unit. Thus, if $p = ab$ then either 'a' is a unit or b is a unit; that is, p is irreducible.

Lemma 45.12. (Continued)

Lemma 45.12. An ideal $\langle p \rangle$ in a PID is maximal if and only if p is irreducible.

Proof. (Continued) Conversely, suppose that p is an irreducible in D. If $\langle p \rangle \subseteq \langle a \rangle$ then by Note 1 Part(1) we must have $p = ab$ for some b in D. If 'a' is a uit, then 'a' and 1 are associates and by Note 1 Part(2), we have $\langle a \rangle = \langle 1 \rangle = D$ and $\langle a \rangle$ is a maximal ideal. If 'a' is not a unit, then b must be a unit (since p is irreducible) so there exists $u \in D$ such that $bu = 1$. Then $pu = abu = a$ and by Note 1 Part(1) $\langle a \rangle \subseteq \langle p \rangle$ and since p and 'a' are associates, by Note 1 Part (2) we have $\langle a \rangle = \langle p \rangle$. We have now shown that if $\langle p \rangle \subseteq \langle a \rangle$ then either $\langle a \rangle = D$ (if 'a' is a unit) or $\langle a \rangle = \langle p \rangle$ (if 'a' is not a unit).

Lemma 45.12. (Continued)

Lemma 45.12. An ideal $\langle p \rangle$ in a PID is maximal if and only if p is irreducible.

Proof. (Continued) So there is no proper ideal of D which properly contains $\langle p \rangle$ (of course all ideals of D are principal). That is, $\langle p \rangle$ is a maximal ideal. □

Lemma 45.13.

Lemma 45.13. In a PID, if an irreducible p divies ab then either $p \mid a$ or $p \mid b$.

Proof. Let D be a PID and suppose that for an irreducible $p \in D$ we have $p \mid ab$. Then $ab \in \langle p \rangle$ (since $\langle p \rangle$ consists of all multiples of p). Since p is irreducible, by Lemma 45.12 $\langle p \rangle$ is a maximal ideal in D. By Corollary 27.16, every maximal ideal is a prime ideal, so $\langle p \rangle$ is a prime ideal. Then $ab \in \langle p \rangle$ implies that either $a \in \langle p \rangle$ or $b \in \langle p \rangle$. That is, by Note 1 Part (1), either $p \mid a$ or $p \mid b$. □

Theorem 45.17.

Theorem 45.17. Every PID is a UFD

Proof. Theorem 45.11 shows that every PID satisfies the first property of a UFD and gives for a in a PID D where 'a' is neither 0 nor a unit, a factorization $a = p_1 p_2 \cdots p_r$ into irreducibles. property 2 of a UFD says that such a factorization is unique (in terms of associates). Let $a = q_1 q_2 \cdots q_s$ be another factorization of 'a' into irreducibles. Then we have $p_1 \mid (q_1 q_2 \cdots q_s)$. By Corollary 45.14, $p_1 \mid q_j$ for some j. Reorder the q's such that q_j becomes q_1. Then $q_1 = p_1 u_1$ where u_1 is a unit. Then p_1 and q_1 are associates. Then $p_1 p_2 \cdots p_r = (p_1 u_1) q_1 q_2 \cdots q_s$. By cancellation in integral domain D (Theorem 19.5) $p_2 p_3 \cdots p_r = u_1 q_1 q_2 \cdots q_s$.

Theorem 45.17. (Continued)

Theorem 45.17. Every PID is a UFD

Proof. (Continued) Repeating the process we have $1 = u_1 u_2 \cdots u_r q_{r+1} \cdots q_s$ (WLOG $s \geq r$). But if $s > R$ and we have some q still present on the right-hand side, say q_{r+1}, then the other elements of the right-hand side are an inverse of the q, for example $(q_{r+1})^{-1} = u_1 u_2 \cdots u_r q_{r+2} q_{r+3} \cdots q_s$. But this contradicts the fact that the q's are irreducible and so (by definition) not units. So there are no q's remaining on the right-hand side and $r = s$. So $p_i = u_i q_i$ for $i = 1, 2, \ldots, r$ and such p_i is an associate of q_i. This is Property 2 in the definition of a UFD and so D is a UFD. □

Corollary 45.18. Fundamental Theorem of Arithmetic

Corollary 45.18. Fundamental Theorem of Arithmetic. The integral domain $\mathbb{Z}$ is a UFD.

Proof. We know that $\mathbb{Z}$ is a PID (see the note after Definition 45.7). So by Theorem 45.17, $\mathbb{Z}$ is a UFD. □

Lemma 45.23.

Lemma 45.23. If D is a UFD then for every nonconstant $f(x) \in D[x]$ we have $f(x) = cg(x)$ where $c \in D$, $g(x) \in D[x]$ and $g(x)$ is a primitive. The element c is unique up to a unit factor in D and is the content of $f(x)$. Also $g(x)$ is unique up to a unit factor in D.

Proof. Let $f(x) \in D[x]$ be given where $f(x)$ is a nonconstant polynomial with coefficients $a_0, a_1, \ldots, a_n$. Let c be a gcd of the a_i. Then for each i, we have $a_i = cg_i$ for some $g_i \in D$. We have $f(x) = cg(x)$. Now there is no irreducible dividing all of the g_i (if so, say the irreducible in b, then cb divides all a_i, but $cb \nmid c$ so in this case c is not a gcd of the a_i). So a gcd of the g_i must be a unit and have an associate of 1. So 1 is a gcd of the g_i and $g(x)$ is a primitive polynomial.

Lemma 45.23. (Continued)

Lemma 45.23. If D is a UFD then for every nonconstant $f(x) \in D[x]$ we have $f(x) = cg(x)$ where $c \in D$, $g(x) \in D[x]$ and $g(x)$ is a primitive. The element c is unique up to a unit factor in D and is the content of $f(x)$. Also $g(x)$ is unique up to a unit factor in D.

Proof. (Continued) For uniqueness, if $f(x) = dh(x)$ also for some $h \in D$ and $h(x) \in D[x]$ with $h(x)$ primitive, then each irreducible factor of c must divide d and each irreducible factor of d must divide c (or else, as in the first paragraph, 1 is not a gcd of the respective coefficients of g or h and hence g or h is not primitive).
By setting $cg(x) = dh(x)$ (since both equal $f(x)$) and cancelling irreducible factors of c into d (Theorem 19.5), we arrive at $ug(x) = vh(x)$ for a unit $u \in D$. But then v must be a unit of D or we would be able to cancel irreducible factors of v into u.

Lemma 45.23. (Continued)

Lemma 45.23. If D is a UFD then for every nonconstant $f(x) \in D[x]$ we have $f(x) = cg(x)$ where $c \in D$, $g(x) \in D[x]$ and $g(x)$ is a primitive. The element c is unique up to a unit factor in D and is the content of $f(x)$. Also $g(x)$ is unique up to a unit factor in D.

Proof. (Continued) So u and v are both units and c is unique up to a unit factor (here, $d = v^{-1}uc$). Since $f(x) = cg(x)$, then the primitive polynomial $g(x)$ is also unique up to a unit factor. □

Lemma 45.25. Gauss's Lemma

Lemma 45.25. Gauss's Lemma. If D is a UFD, then a product of two primitive polynomials in $D[x]$ is again primitive.

Proof. Let $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ and $g(x) = b_0 + b_1x + b_2x^2 + \ldots + b_mx^m$ be primitives in $D[x]$ and let $h(x) = f(x)g(x)$. Let p be an irreducible in D. Then p does not divide all a_i and p does not divide b_j* (or else a multiple of p is a gcd of the a_i and of the b_j and 1 is not a gcd since all gcd's are associates). [since $f(x)$ and $g(x)$ are primitive.]
Let a_r be the first coefficient (i.e., r is the smallest value) of $f(x)$ not divisible by p; that is, $p \mid a_i$ for $0 \le i < r$ but $p \nmid a_r$. Similarly, let $p \mid b_j$ for $0 \le i < s$ but $p \nmid b_s$.

Lemma 45.25 Gauss's Lemma. (Continued)

Lemma 45.25 Gauss's Lemma. If D is a UFD, then a product of two primitive polynomials in $D[x]$ is again primitive.

Proof. (Continued) The cofficient of x^{r+s} in $h(x) = f(x)g(x)$ is (we are in a commutative ring):

$$\begin{aligned} c_{r+s} = &(a_0 b_{r+s} + a_1 b_{r+s-1} + \ldots + a_{r-1} b_{s+1}) + a_r b_s \\ &+ (a_{r+1} b_{s-1} + a_{r+2} b_{s-2} + \ldots + a_{r+s} b_0) \end{aligned} \quad (1)$$

Now $p \mid a_i$ for $0 \le i < r$ implies that $p \mid (a_0 b_{r+s} + a_1 b_{r+s-1} + \ldots + a_{r-1} b_{s+1})$ and $p \mid b_j$ for $0 \le j < s$ implies that $p \mid (a_{r+1} b_{s-1} + a_{r+2} b_{s-2} + \ldots + a_{r+s} b_0)$.

Lemma 45.25 Gauss's Lemma. (Continued)

Lemma 45.25 Gauss's Lemma. If D is a UFD, then a product of two primitive polynomials in $D[x]$ is again primitive.

Proof. (Continued) But p does not divide a_r or b_s, so p does not divide $a_r b_s$ and consequently p does not divide c_{r+s}. So we have that any irreducible $p \in D$ does not divide some coefficient of $f(x)g(x)$. So the gcd of the coefficients of $f(x)g(x)$ is 1 and $f(x)g(x)$ is primitive. □

Lemma 45.27.

Lemma 45.27. Let D be a UFD and let F be a field of quotients of D. Let $f(x) \in D[x]$ where $(\textit{degree}\, f(x)) > 0$. If $f(x)$ is an irreducible in $D[x]$, then $f(x)$ is also an irreducible in $F[x]$. Also, if $f(x)$ is primitive in $D[x]$ and irreducible in $F[x]$, then $f(x)$ is irreducible in $D[x]$.

Proof. We prove the contrapositive of the first claim. Suppose that a nonconstant $f(x) \in D[x]$ factors into polynomials of lower degree in $F[x]$; that is $f(x) = r(x)s(x)$ for $r(x), s(x) \in F[x]$. Then since F is a field of quotients of D, each coefficient in $r(x)$ and $s(x)$ is of the form a/b for some $a, b \in D$, $b \neq 0$. By "clearing the denominators" (i.e. multiplying through by a common multiple of the denominator) we can get $df(x) = r_1(x)s_1(x)$ for $d \in D$ and $r_1(x), s_1(x) \in D[x]$ where the degrees of $r_1(x)$ and $s_1(x)$ equal the degrees of $r(x)$ and $s(x)$, respectively.

Lemma 45.27. (Continued)

Lemma 45.27. Let D be a UFD and let F be a field of quotients of D. Let $f(x) \in D[x]$ where $(\textit{degree}\, f(x)) > 0$. If $f(x)$ is an irreducible in $D[x]$, then $f(x)$ is also an irreducible in $F[x]$. Also, if $f(x)$ is primitive in $D[x]$ and irreducible in $F[x]$, then $f(x)$ is irreducible in $D[x]$.

Proof. (Continued) By Lemma 45.23 $f(x) = cg(x)$, $r_1(x) = c_1 r_2(x)$, and $s_1(x) = c_2 s_2(x)$ for primitive polynomials $g(x), r_2(x)$ and $s_2(x)$ in $D[x]$ and $c, c_1, c_2 \in D$. Then $dcg(x) = c_1 r_2(x) c_2 s_2(x) = c_1 c_2 r_2(x) s_2(x)$ and by Lemma 45.25 the product $r_2(x)s_2(x)$ is primitive. By the uniqueness part of Lemma 45.23, $c_1 c_2 = dcu$ for some unit u in D. But then $dcg(x) = dcur_2(x)s_2(x)$ and so $f(x) = cg(x) = cur_2(x)s_2(x)$ where $cu \in D$ and $r_2(x), s_2(x) \in D[x]$.
So $f(x)$ factors nontrivially into polynomials of the same degree in $D[x]$ as the degree of the polynomial factors of $f(x)$ in $F[x]$.
A nonconstant $f(x) \in D[x]$ that is primitive in $D[x]$ and irreducible in $F[x]$ is also irreducible in $D[x]$ since $D[x] \subseteq F[x]$. □

Corollary 45.28.

Corollary 45.28. If D is a UFD and F is a field of quotients of D, then a nonconstant $f(x) \in D[x]$ factors into a product of two polynomials of lower degrees r and s in $F[x]$ if and only if it has a factorization into polynomials of the same degrees r and s in $D[x]$.

Proof. In the proof of Lemma 45.27, if $f(x)$ factors in $F[x]$ into $f(x) = r(x)s(x)$ where $r(x)$ and $s(x)$ are of degrees smaller than the degree of $f(x)$, then $f(x) = cur_2(x)s_2(x)$ in $D[x]$ where the degrees of $r(x)$ and $r_2(x)$ are the same and the degrees of $s(x)$ and $s_2(x)$ are the same. The converse holds since $D[x] \subseteq F[x]$. □

Theorem 45.29.

Theorem 45.29. If D is a UFD, then $D[x]$ is a UFD.

Proof. Let $f(x) \in D[x]$ where $f(x)$ is neither 0 nor a unit. If $f(x)$ is of degree 0, we are done since D is a UFD. Suppose $(degree\, f(x)) > 0$. Let $f(x) = g_1(x)g_2(x) \cdots g_r(x)$ be a factorization of $f(x)$ in $D[x]$ having the greatest number r of factors of positive degree (so no $g_i(x)$ is a constant polynomial). There is such a greatest number of such factors since r cannot exceed the degree of $f(x)$.

Theorem 45.29. (Continued)

Theorem 45.29. If D is a UFD, then $D[x]$ is a UFD.

Proof. (Continued) Now factor each $g_i(x)$ in the form $g_i(x) = c_i h(x)$ where c_i is the content of $g_i(x)$ (by Lemma 45.23, c is a gcd of the coefficients of $g_i(x)$) and $h_i(x)$ is a primitive polynomial. Also, each $h_i(x)$ must be irreducible; if an $h_i(x)$ could be factored then the corresponding factorization of $g_i(x)$ (described in the proof of Lemma 45.27) would give a factorization of $f(x)$ with more than r factors, contradicting the choice of r. Thus we now have $f(x) = c_1 h_1(x) c_2 h_2(x) \cdots c_r h_r(x)$ where the $h_i(x)$ are irreducible in $D[x]$. If we now factor the c_i into irreducibles in D (since D is a UFD), we obtain a factorization of $f(x)$ into a product of irreducibles in $D[x]$.

Theorem 45.29. (Continued)

Theorem 45.29. If D is a UFD, then $D[x]$ is a UFD.

Proof. (Continued) The factorization of $f(x) \in D[x]$ where $f(x)$ has degree 0 is unique since D is a UFD. If $f(x)$ has degree greater than 0, then any factorization of $f(x)$ into irreducibles in $D[x]$ corresponds to a factorization in $F[x]$ into units (the factors in D; the constant factors) and, by Lemma 45.27, irreducible polynomials in $F[x]$. By Theorem 23.20, these irreducible polynomials are unique, except for possible constant factors in F. But as an irreducible in $D[x]$, each polynomial of degree > 0 appearing in the factorization of $f(x)$ in $D[x]$ is primitive (or else the constant gcd of the coefficients could be factored out).

Theorem 45.29. (Continued)

Theorem 45.29. If D is a UFD, then $D[x]$ is a UFD.

Proof. (Continued) By the uniqueness part of Lemma 45.23, these irreducible polynomial factors are unique in $D[x]$ up to unit factors (that is, unique up to being associates). The product of the irreducibles in D in the factorization of $f(x)$ (that is, the constant factors) is the content of $f(x)$, which is unique up to a unit facotr by Lemma 45.23. Thus all irreducibles in $D[x]$ appearing in the factorization are unique up to order and associates. □

Corollary 45.30.

Corollary 45.30. If F is a field and x_1, x_2, ..., x_n are indeterminates, then $F[x_1, x_2, ..., x_n]$ is a UFD.

Proof. By Theorem 23.20, $F[x]$ is a UFD. By Corollary 45.30 and induction, $F[x_1, x_2]$, $F[x_1, x_2, x_3]$,...,$F[x_1, x_2, ..., x_n]$ are UFDs. □