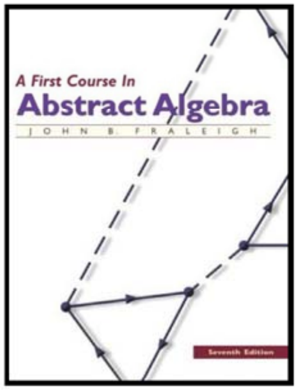
Introduction to Modern Algebra

Part VII. Advanced Group Theory

VII.39. Free Groups



Theorem 39.12.

Theorem. 39.12. Let G be a group generated by $A = \{a_i \mid i \in I\}$ and let G' be any group. If a'_i for $i \in I$ are any elements in G', not necessarily distinct, then there is at most one homomorphism $\phi :\to G'$ such that $\phi(a_i) = a'_i$. If G is free on A, then there exists exactly one such homomorphism.

Proof. Suppose ϕ is a homomorphism from G into G' such that $\phi(a_i) = a'_i$ (we show that there is not a second such homomorphism). Since A is a generating set for G, then by Theorem 7.6, for any $x \in G$ we have $x = \prod_{j \in J} a_{i_j}^{n_j}$ for some finite set of indices J, where the a_{i_j} need not be distinct. Since ϕ is a homomorphism then

$$\begin{aligned} \phi(x) = \phi(\prod_{j \in J} a_{i_j}^{n_j}) &= \prod_{j \in J} \phi(a_{i_j}^{n_j}) \\ = \prod_{j \in J} \phi(a_{i_j})^{n_j} &= \prod_{j \in J} (a'_{i_j})^{n_j} \end{aligned} \qquad (1)$$

Theorem 39.12. (Continued)

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Proof (Continued). So homomorphism ϕ is completely determined by its values on the elements of set A. Therefore there is at most one homomorphism mapping a_i to a'_i, $i \in I$.
Now suppose G is free on A; that is $G = F[A]$. For $x = \prod_{j \in J} a_{i_j}^{n_j} \in G$, define $\psi : G \to G'$ by $\psi(x) = \prod_{j \in J} (a'_{i_j})^{n_j}$. (Notice that ψ is well defined since $G = F[A]$ consists only of reduced words and so different products of the form of x yield different elements of $F[A]$.)

Theorem 39.12. (Continued)

Theorem. 39.12. Let G be a group generated by $A = \{a_i \mid i \in I\}$ and let G' be any group. If a'_i for $i \in I$ are any elements in G', not necessarily distinct, then there is at most one homomorphism $\phi :\to G'$ such that $\phi(a_i) = a'_i$. If G is free on A, then there exists exactly one such homomorphism.

Proof (Continued). Since the rules for computation involving exponents in G' are formally the same as those involving exponents in G' (that is, the elementary contractions on the a_{i_j} in G exactly correspond to the elementary contractions on the a'_{i_j} in G'). So $\psi(xy) = \psi(x)\psi(y)$ for all $x, y \in G$ and ψ is a homomorphism. □

Theorem 39.13.

Theorem. 39.13. Every group G' is a homomorphic image of a free group G.

Proof. Let $G' = \{a'_i \mid i \in I\}$, and let $A = \{a_i \mid i \in I\}$ be a set with the same number of elements as G'. Let $G = F[A]$ (so G is the free group generated by set A). Since G is free, by Theorem 39.12 there exists a homomorphism ψ mapping G into G' such that $\psi(a_i) = a'_i$ for $i \in I$. Since $|G'| = |A'|$, then ψ is onto G' and $\psi[G] = G'$. □

Gallian's "Universal Quotient Group Property."

Theorem. Gallian's "Universal Quotient Group Property." Every group is isomorphic to a quotient group of a free group.

Proof. Let G' be a group. By Theorem 39.13, there is a free group G and a homomorphism ψ such that $\psi[G] = G'$. Let $K = Ker(\psi)$. Then by the First Isomorphism Theorem (Theorem 34.2), there is a unique isomorphism $\mu : G/K \to \psi[G]$. So μ is an isomorphism from the quotient group G/K of the free group G to group G', $\mu : G/K \to G'$. □