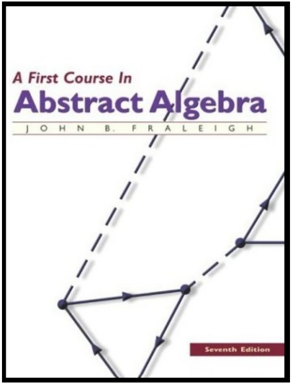
Introduction to Modern Algebra

Part VII. Advanced Group Theory

VII.39. Free Groups



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$$\begin{aligned} \phi(x) = \phi(\prod_{j \in J} a_{i_j}^{n_j}) &= \prod_{j \in J} \phi(a_{i_j}^{n_j}) \\ &= \prod_{j \in J} \phi(a_{i_j})^{n_j} = \prod_{j \in J} (a_{i_j}')^{n_j} \end{aligned} \tag{1}$$

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Proof (Continued). So homomorphism ϕ is completely determined by its values on the elements of set A. Therefore there is at most one homomorphism mapping a_i to a'_i, $i \in I$.
Now suppose G is free on A; that is $G = F[A]$. For $x = \prod_{j \in J} a_{i_j}^{n_j} \in G$, define $\psi : G \rightarrow G'$ by $\psi(x) = \prod_{j \in J}(a'_{i_j})^{n_j}$. (Notice that ψ is well defined since $G = F[A]$ consists only of reduced words and so different products of the form of x yield different elements of $F[A]$.)

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Proof. Let $G' = \{a'_i \mid i \in I\}$, and let $A = \{a_i \mid i \in I\}$ be a set with the same number of elements as G'. Let $G = F[A]$ (so G is the free group generated by set A).

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Gallian's "Universal Quotient Group Property."

Theorem. Gallian's "Universal Quotient Group Property." Every group is isomorphic to a quotient group of a free group.

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