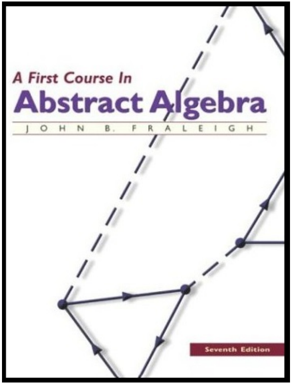
# Introduction to Modern Algebra

**Part X. Automorphisms and Galois Theory**

X.49. The Isomorphism Extension Theorem



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# Theorem 49.3

**Theorem 49.3.** Let $E$ be an algebraic extension of a field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$. Let $\bar{F}'$ be an algebraic closure of $F'$. Then $\sigma$ can be extended to an isomorphism $\tau$ of $E$ onto a subfield $\bar{F}'$ such that $\tau(a) = \sigma(a)$ for all $a \in F$.

**Proof.** Consider all pairs $(L, \lambda)$ where $L$ is a field such that $F \leq L \leq E$ and $\lambda$ is an isomorphism of $L$ onto a subfield of $\bar{F}'$ such that $\lambda(a) = \sigma(a)$ for all $a \in F$. Let $S$ be the set of all such pairs. $S$ is nonempty since $(F, \sigma) \in S$.

# Theorem 49.3

**Theorem 49.3.** Let $E$ be an algebraic extension of a field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$. Let $\bar{F}'$ be an algebraic closure of $F'$. Then $\sigma$ can be extended to an isomorphism $\tau$ of $E$ onto a subfield $\bar{F}'$ such that $\tau(a) = \sigma(a)$ for all $a \in F$.

**Proof.** Consider all pairs $(L, \lambda)$ where $L$ is a field such that $F \leq L \leq E$ and $\lambda$ is an isomorphism of $L$ onto a subfield of $\bar{F}'$ such that $\lambda(a) = \sigma(a)$ for all $a \in F$. Let $S$ be the set of all such pairs. $S$ is nonempty since $(F, \sigma) \in S$. Define $\leq$ on $S$ as $(L_1, \lambda_1) \leq (L_2, \lambda_2)$ if $L_1 \leq L_2$ and $\lambda_1(a)\lambda_2(a)$ for all $a \in L_1$ (in which case $\lambda_2$ extends $\lambda_1$ from $L_1$ to $L_2$).

# Theorem 49.3

**Theorem 49.3.** Let $E$ be an algebraic extension of a field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$. Let $\bar{F}'$ be an algebraic closure of $F'$. Then $\sigma$ can be extended to an isomorphism $\tau$ of $E$ onto a subfield $\bar{F}'$ such that $\tau(a) = \sigma(a)$ for all $a \in F$.

**Proof.** Consider all pairs $(L, \lambda)$ where $L$ is a field such that $F \leq L \leq E$ and $\lambda$ is an isomorphism of $L$ onto a subfield of $\bar{F}'$ such that $\lambda(a) = \sigma(a)$ for all $a \in F$. Let $S$ be the set of all such pairs. $S$ is nonempty since $(F, \sigma) \in S$. Define $\leq$ on $S$ as $(L_1, \lambda_1) \leq (L_2, \lambda_2)$ if $L_1 \leq L_2$ and $\lambda_1(a)\lambda_2(a)$ for all $a \in L_1$ (in which case $\lambda_2$ extends $\lambda_1$ from $L_1$ to $L_2$).

# Theorem 49.3 (continued 1)

**Theorem 49.3.** Let $E$ be an algebraic extension of a field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$. Let $\bar{F}'$ be an algebraic closure of $F'$. Then $\sigma$ can be extended to an isomorphism $\tau$ of $E$ onto a subfield $\bar{F}'$ such that $\tau(a) = \sigma(a)$ for all $a \in F$.

**Proof (continued).** Claim $\leq$ is a partial ordering of $S$.
For any $(L, \lambda) \in S$ we have $L \leq L$ and $\lambda(a) = \sigma(a)$ for all $a \in F \leq L$, so $(L, \lambda) \leq (L, \lambda)$ and $\leq$ is reflexive. If $(L_1, \lambda_1) \leq (L_2, \lambda_2)$ and $(L_2, \lambda_2) \leq (L_1, \lambda_1)$ then $L_1 \leq L_2$ and $L_2 \leq L_1$, so $L_1 = L_2$.

# Theorem 49.3 (continued 1)

**Theorem 49.3.** Let $E$ be an algebraic extension of a field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$. Let $\bar{F}'$ be an algebraic closure of $F'$. Then $\sigma$ can be extended to an isomorphism $\tau$ of $E$ onto a subfield $\bar{F}'$ such that $\tau(a) = \sigma(a)$ for all $a \in F$.

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For any $(L, \lambda) \in S$ we have $L \leq L$ and $\lambda(a) = \sigma(a)$ for all $a \in F \leq L$, so $(L, \lambda) \leq (L, \lambda)$ and $\leq$ is reflexive. If $(L_1, \lambda_1) \leq (L_2, \lambda_2)$ and $(L_2, \lambda_2) \leq (L_1, \lambda_1)$ then $L_1 \leq L_2$ and $L_2 \leq L_1$, so $L_1 = L_2$. Also, $\lambda_1(a) = \lambda_2(a)$ for all $a \in F$. So $(L_1, \lambda_1) = (L_2, \lambda_2)$ and $\leq$ is antisymmetric.

# Theorem 49.3 (continued 1)

**Theorem 49.3.** Let $E$ be an algebraic extension of a field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$. Let $\bar{F}'$ be an algebraic closure of $F'$. Then $\sigma$ can be extended to an isomorphism $\tau$ of $E$ onto a subfield $\bar{F}'$ such that $\tau(a) = \sigma(a)$ for all $a \in F$.

**Proof (continued).** Claim $\leq$ is a partial ordering of $S$.
For any $(L, \lambda) \in S$ we have $L \leq L$ and $\lambda(a) = \sigma(a)$ for all $a \in F \leq L$, so $(L, \lambda) \leq (L, \lambda)$ and $\leq$ is reflexive. If $(L_1, \lambda_1) \leq (L_2, \lambda_2)$ and $(L_2, \lambda_2) \leq (L_1, \lambda_1)$ then $L_1 \leq L_2$ and $L_2 \leq L_1$, so $L_1 = L_2$. Also, $\lambda_1(a) = \lambda_2(a)$ for all $a \in F$. So $(L_1, \lambda_1) = (L_2, \lambda_2)$ and $\leq$ is antisymmetric.
Suppose $(L_1, \lambda_1) \leq (L_2, \lambda_2)$ and $(L_2, \lambda_2) \leq (L_3, \lambda_3)$. Then $L_1 \leq L_2 \leq L_3$ and so $L_1 \leq L_3$. Also, $\lambda_1(a) = \lambda_2(a) = \lambda_3(a)$ for all $a \in F$. So $(L_1, \lambda_1) \leq (L_3, \lambda_3)$ and $\leq$ is transitive.

# Theorem 49.3 (continued 1)

**Theorem 49.3.** Let $E$ be an algebraic extension of a field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$. Let $\bar{F}'$ be an algebraic closure of $F'$. Then $\sigma$ can be extended to an isomorphism $\tau$ of $E$ onto a subfield $\bar{F}'$ such that $\tau(a) = \sigma(a)$ for all $a \in F$.

**Proof (continued).** Claim $\leq$ is a partial ordering of $S$.
For any $(L, \lambda) \in S$ we have $L \leq L$ and $\lambda(a) = \sigma(a)$ for all $a \in F \leq L$, so $(L, \lambda) \leq (L, \lambda)$ and $\leq$ is reflexive. If $(L_1, \lambda_1) \leq (L_2, \lambda_2)$ and $(L_2, \lambda_2) \leq (L_1, \lambda_1)$ then $L_1 \leq L_2$ and $L_2 \leq L_1$, so $L_1 = L_2$. Also, $\lambda_1(a) = \lambda_2(a)$ for all $a \in F$. So $(L_1, \lambda_1) = (L_2, \lambda_2)$ and $\leq$ is antisymmetric.
Suppose $(L_1, \lambda_1) \leq (L_2, \lambda_2)$ and $(L_2, \lambda_2) \leq (L_3, \lambda_3)$. Then $L_1 \leq L_2 \leq L_3$ and so $L_1 \leq L_3$. Also, $\lambda_1(a) = \lambda_2(a) = \lambda_3(a)$ for all $a \in F$. So $(L_1, \lambda_1) \leq (L_3, \lambda_3)$ and $\leq$ is transitive.

# Theorem 49.3 (continued 2)

**Theorem 49.3.** Let $E$ be an algebraic extension of a field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$. Let $\bar{F}'$ be an algebraic closure of $F'$. Then $\sigma$ can be extended to an isomorphism $\tau$ of $E$ onto a subfield $\bar{F}'$ such that $\tau(a) = \sigma(a)$ for all $a \in F$.

**Proof (continued).** Let $T = \{(H_i, \lambda_i) \mid i \in I\}$ be a chain in $S$. Claim $H = \bigcup_{i \in I} H_i$ is a subfield of $E$.
Let $a, b \in H$ where $a \in H_i$ and $b \in H_j$. Then either $H_i \leq H_j$ or $H_j \leq H_i$ since $T$ is a chain (definition of "chain"). WLOG, say $H_i \leq H_j$ then $a, b \in H_j$, so $a \pm b$, $ab$, and $a/b$ for $b \neq 0$ are in $H_j$ since $H_j$ is a field.

# Theorem 49.3 (continued 2)

**Theorem 49.3.** Let $E$ be an algebraic extension of a field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$. Let $\bar{F}'$ be an algebraic closure of $F'$. Then $\sigma$ can be extended to an isomorphism $\tau$ of $E$ onto a subfield $\bar{F}'$ such that $\tau(a) = \sigma(a)$ for all $a \in F$.

**Proof (continued).** Let $T = \{(H_i, \lambda_i) \mid i \in I\}$ be a chain in $S$. Claim $H = \bigcup_{i \in I} H_i$ is a subfield of $E$.
Let $a, b \in H$ where $a \in H_i$ and $b \in H_j$. Then either $H_i \leq H_j$ or $H_j \leq H_i$ since $T$ is a chain (definition of "chain"). WLOG, say $H_i \leq H_j$ then $a, b \in H_j$, so $a \pm b$, $ab$, and $a/b$ for $b \neq 0$ are in $H_j$ since $H_j$ is a field. Since $H_j \subseteq H$, then $a \pm b$, $ab$, $a/b \in H$. Therefore, $H$ is a field. Since for each $i \in I$ we have $F \subseteq H_i \subseteq E$ then $F \subseteq H \subseteq E$. So $H$ is a subfield of $E$.

# Theorem 49.3 (continued 2)

**Theorem 49.3.** Let $E$ be an algebraic extension of a field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$. Let $\bar{F}'$ be an algebraic closure of $F'$. Then $\sigma$ can be extended to an isomorphism $\tau$ of $E$ onto a subfield $\bar{F}'$ such that $\tau(a) = \sigma(a)$ for all $a \in F$.

**Proof (continued).** Let $T = \{(H_i, \lambda_i) \mid i \in I\}$ be a chain in $S$. Claim $H = \bigcup_{i \in I} H_i$ is a subfield of $E$.

Let $a, b \in H$ where $a \in H_i$ and $b \in H_j$. Then either $H_i \leq H_j$ or $H_j \leq H_i$ since $T$ is a chain (definition of "chain"). WLOG, say $H_i \leq H_j$ then $a, b \in H_j$, so $a \pm b$, $ab$, and $a/b$ for $b \neq 0$ are in $H_j$ since $H_j$ is a field. Since $H_j \subseteq H$, then $a \pm b$, $ab$, $a/b \in H$. Therefore, $H$ is a field. Since for each $i \in I$ we have $F \subseteq H_i \subseteq E$ then $F \subseteq H \subseteq E$. So $H$ is a subfield of $E$.

# Theorem 49.3. (continued 3)

**Theorem 49.3.** Let $E$ be an algebraic extension of a field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$. Let $\bar{F}'$ be an algebraic closure of $F'$. Then $\sigma$ can be extended to an isomorphism $\tau$ of $E$ onto a subfield $\bar{F}'$ such that $\tau(a) = \sigma(a)$ for all $a \in F$.

**Proof (continued).** Define $\lambda : H \to \bar{F}'$ as $\lambda(c) = \lambda_i(c)$ for each $c \in H$ where $c \in H_i$. We need to show that $\lambda$ is well defined (and hence independent of the choice of $H_i$). Notice that if $c \in H_i$ and $c \in H_j$ then either $(H_i, \lambda_i) \leq (H_j, \lambda_j)$ or $(H_j, \lambda_j) \leq (H_i, \lambda_i)$ since $T$ is a chain. In either case $\lambda_i(c) = \lambda_j(c)$ and so $\lambda(c)$ is well defined.

# Theorem 49.3. (continued 3)

**Theorem 49.3.** Let $E$ be an algebraic extension of a field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$. Let $\bar{F}'$ be an algebraic closure of $F'$. Then $\sigma$ can be extended to an isomorphism $\tau$ of $E$ onto a subfield $\bar{F}'$ such that $\tau(a) = \sigma(a)$ for all $a \in F$.

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# Theorem 49.3 (continued 4)

**Theorem 49.3.** Let $E$ be an algebraic extension of a field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$. Let $\bar{F}'$ be an algebraic closure of $F'$. Then $\sigma$ can be extended to an isomorphism $\tau$ of $E$ onto a subfield $\bar{F}'$ such that $\tau(a) = \sigma(a)$ for all $a \in F$.

**Proof (continued).** Claim $\lambda : H \to \bar{F}'$ is an isomorphism of $H$ onto a subfield $\lambda[H]$ of $\bar{F}'$. If $a, b \in H$ then there is (as above) an $H_j$ such that $a, b \in H_j$ and

$$\begin{aligned} \lambda(a+b) = \lambda_j(a+b) &= \lambda_j(a) + \lambda(b) \text{ since } \lambda_j \text{ is an isomorphism} \\ &= \lambda(a) + \lambda(b) \\ \lambda(ab) = \lambda_j(ab) &= \lambda_j(a)\lambda(b) \text{ since} \lambda_j \text{ is an isomorphism} \\ &= \lambda(a)\lambda(b) \end{aligned} \quad (1)$$

So $\lambda$ has the homomorphism property with respect to addition and multiplication.

# Theorem 49.3 (continued 4)

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# Theorem 49.3 (continued 5)

**Theorem 49.3.** Let $E$ be an algebraic extension of a field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$. Let $\bar{F}'$ be an algebraic closure of $F'$. Then $\sigma$ can be extended to an isomorphism $\tau$ of $E$ onto a subfield $\bar{F}'$ such that $\tau(a) = \sigma(a)$ for all $a \in F$.

**Proof (continued).** Next, if $\lambda(a) = 0$ for $a \in H$, then $a \in H_i$ for some $i \in I$ and so $\lambda_i(a) = \lambda(a) = 0$ implies that $a = 0$ since $\lambda_i$ is an isomorphism and hence one to one (by Corollary 13.18). Therefore (by Corollary 13.18) $Ker(\lambda) = 0$ and $\lambda$ is onto $\lambda[H]$ and therefore $\lambda$ is an isomorphism with $\lambda[H]$ a subfield of $\bar{F}'$.

Since each $\lambda_i$ fixes $F$, then $\lambda$ fixes $F$. So $(H, \lambda) \in S$. By construction (since $H_i \subseteq H$ for all $i \in I$), $(H, \lambda)$ is an upper bound for chain $T$.

# Theorem 49.3 (continued 5)

**Theorem 49.3.** Let $E$ be an algebraic extension of a field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$. Let $\bar{F}'$ be an algebraic closure of $F'$. Then $\sigma$ can be extended to an isomorphism $\tau$ of $E$ onto a subfield $\bar{F}'$ such that $\tau(a) = \sigma(a)$ for all $a \in F$.

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Since each $\lambda_i$ fixes $F$, then $\lambda$ fixes $F$. So $(H, \lambda) \in S$. By construction (since $H_i \subseteq H$ for all $i \in I$), $(H, \lambda)$ is an upper bound for chain $T$. Since $T$ was an arbitrary chain, then every chain in $S$ has an upper bound in $S$. So $S$ satisfies the hypotheses of Zorn's Lemma.

# Theorem 49.3 (continued 5)

**Theorem 49.3.** Let $E$ be an algebraic extension of a field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$. Let $\bar{F}'$ be an algebraic closure of $F'$. Then $\sigma$ can be extended to an isomorphism $\tau$ of $E$ onto a subfield $\bar{F}'$ such that $\tau(a) = \sigma(a)$ for all $a \in F$.

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Since each $\lambda_i$ fixes $F$, then $\lambda$ fixes $F$. So $(H, \lambda) \in S$. By construction (since $H_i \subseteq H$ for all $i \in I$), $(H, \lambda)$ is an upper bound for chain $T$. Since $T$ was an arbitrary chain, then every chain in $S$ has an upper bound in $S$. So $S$ satisfies the hypotheses of Zorn's Lemma.

# Theorem 49.3 (continued 6)

**Theorem 49.3.** Let $E$ be an algebraic extension of a field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$. Let $\bar{F}'$ be an algebraic closure of $F'$. Then $\sigma$ can be extended to an isomorphism $\tau$ of $E$ onto a subfield $\bar{F}'$ such that $\tau(a) = \sigma(a)$ for all $a \in F$.

**Proof. (Continued)** Applying Zorn's Lemma, there is a maximal element of $S$, say $(K, \tau)$. Denote $\tau[K]$ as $K'$ and so $K' \leq \bar{F}'$. ASSUME $K \neq E$. Let $\alpha \in E/K$. Since $E$ is an algebraic extension of $F$, $\alpha$ is algebraic over $K$. Let $p(x) = \mathrm{irr}(\alpha, K)$.

# Theorem 49.3 (continued 6)

**Theorem 49.3.** Let $E$ be an algebraic extension of a field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$. Let $\bar{F}'$ be an algebraic closure of $F'$. Then $\sigma$ can be extended to an isomorphism $\tau$ of $E$ onto a subfield $\bar{F}'$ such that $\tau(a) = \sigma(a)$ for all $a \in F$.

**Proof. (Continued)** Applying Zorn's Lemma, there is a maximal element of $S$, say $(K, \tau)$. Denote $\tau[K]$ as $K'$ and so $K' \leq \bar{F}'$. ASSUME $K \neq E$. Let $\alpha \in E/K$. Since $E$ is an algebraic extension of $F$, $\alpha$ is algebraic over $K$. Let $p(x) = \mathrm{irr}(\alpha, K)$. Consider the evaluation homomorphism $\varphi_\alpha : K[x] \to K(\alpha)$ (so the symbol $x$ is simply replaced by $\alpha$; see Theorem 22.4, "The Evaluation Homomorphisms for Field Theory").

# Theorem 49.3 (continued 6)

**Theorem 49.3.** Let $E$ be an algebraic extension of a field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$. Let $\bar{F}'$ be an algebraic closure of $F'$. Then $\sigma$ can be extended to an isomorphism $\tau$ of $E$ onto a subfield $\bar{F}'$ such that $\tau(a) = \sigma(a)$ for all $a \in F$.

**Proof. (Continued)** Applying Zorn's Lemma, there is a maximal element of $S$, say $(K, \tau)$. Denote $\tau[K]$ as $K'$ and so $K' \leq \bar{F}'$. ASSUME $K \neq E$. Let $\alpha \in E/K$. Since $E$ is an algebraic extension of $F$, $\alpha$ is algebraic over $K$. Let $p(x) = \text{irr}(\alpha, K)$. Consider the evaluation homomorphism $\varphi_\alpha : K[x] \to K(\alpha)$ (so the symbol $x$ is simply replaced by $\alpha$; see Theorem 22.4, "The Evaluation Homomorphisms for Field Theory").

# Theorem 49.3 (continued 7)

**Theorem 49.3.** Let $E$ be an algebraic extension of a field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$. Let $\bar{F}'$ be an algebraic closure of $F'$. Then $\sigma$ can be extended to an isomorphism $\tau$ of $E$ onto a subfield $\bar{F}'$ such that $\tau(a) = \sigma(a)$ for all $a \in F$.

**Proof (continued).** Let $\psi_\alpha$ be the canonical isomorphism mapping $K[x]/\langle p(x)\rangle$ onto $K(\alpha)$ which corresponds to $\varphi_\alpha$. The elements of $K[x]/\langle p(x)\rangle$ are cosets of $\langle p(x)\rangle$, say of the form $r(x) + \langle p(x)\rangle$. Since $p(\alpha) = 0$, then $\psi_\alpha(r(x) + \langle p(x)\rangle) = r(\alpha) \in K(\alpha)$. Notice that $\psi_\alpha$ is one to one since it maps cosets to elements of $K(\alpha)$ (whereas $\varphi_\alpha$ is not one to one; it maps elements from the same cosets of $\langle p(x)\rangle$ onto the same element of $K(\alpha)$).

# Theorem 49.3 (continued 7)

**Theorem 49.3.** Let $E$ be an algebraic extension of a field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$. Let $\bar{F}'$ be an algebraic closure of $F'$. Then $\sigma$ can be extended to an isomorphism $\tau$ of $E$ onto a subfield $\bar{F}'$ such that $\tau(a) = \sigma(a)$ for all $a \in F$.

**Proof (continued).** Let $\psi_\alpha$ be the canonical isomorphism mapping $K[x]/\langle p(x)\rangle$ onto $K(\alpha)$ which corresponds to $\varphi_\alpha$. The elements of $K[x]/\langle p(x)\rangle$ are cosets of $\langle p(x)\rangle$, say of the form $r(x) + \langle p(x)\rangle$. Since $p(\alpha) = 0$, then $\psi_\alpha(r(x) + \langle p(x)\rangle) = r(\alpha) \in K(\alpha)$. Notice that $\psi_\alpha$ is one to one since it maps cosets to elements of $K(\alpha)$ (whereas $\varphi_\alpha$ is not one to one; it maps elements from the same cosets of $\langle p(x)\rangle$ onto the same element of $K(\alpha)$).

# Theorem 49.3 (continued 8)

**Theorem 49.3.** Let $E$ be an algebraic extension of a field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$. Let $\bar{F}'$ be an algebraic closure of $F'$. Then $\sigma$ can be extended to an isomorphism $\tau$ of $E$ onto a subfield $\bar{F}'$ such that $\tau(a) = \sigma(a)$ for all $a \in F$.

**Proof (continued).** If $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, consider $q(x) = \tau(a_0) + \tau(a_1)x + \tau(a_2)x^2 + \cdots + \tau(a_n)x^n \in K'[x] = \tau[K][x]$ where $\tau$ is from the maximal element $(K, \tau)$ of $S$. Since $\tau$ is an isomorphism, $q(x)$ is irreducible in $K'[x]$. Since $K' \leq \bar{F}'$, there is a zero $\alpha'$ of $q(x)$ in $\bar{F}'$.

# Theorem 49.3 (continued 8)

**Theorem 49.3.** Let $E$ be an algebraic extension of a field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$. Let $\bar{F}'$ be an algebraic closure of $F'$. Then $\sigma$ can be extended to an isomorphism $\tau$ of $E$ onto a subfield $\bar{F}'$ such that $\tau(a) = \sigma(a)$ for all $a \in F$.

**Proof (continued).** If $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, consider $q(x) = \tau(a_0) + \tau(a_1)x + \tau(a_2)x^2 + \cdots + \tau(a_n)x^n \in K'[x] = \tau[K][x]$ where $\tau$ is from the maximal element $(K, \tau)$ of $S$. Since $\tau$ is an isomorphism, $q(x)$ is irreducible in $K'[x]$. Since $K' \leq \bar{F}'$, there is a zero $\alpha'$ of $q(x)$ in $\bar{F}'$. Similar to above, let $\psi_{\alpha'} : K'[x]/\langle q(x)\rangle \to K'(\alpha')$. Finally, let $\bar{\tau} : K[x]/\langle p(x)\rangle \to K'[x]/\langle q(x)\rangle$ be an isomorphism "extending" $\tau$ from $K$ to $K[x]/\langle p(x)\rangle$ (since $K \cong K'$ and $\tau$ "maps" $p(x) \in K[x]$ to $q(x) \in K'[x]$, then such an isomorphic relation exists). We have $\tau(x + \langle p(x)\rangle) = x + \langle q(x)\rangle$, for example.

# Theorem 49.3 (continued 8)

**Theorem 49.3.** Let $E$ be an algebraic extension of a field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$. Let $\bar{F}'$ be an algebraic closure of $F'$. Then $\sigma$ can be extended to an isomorphism $\tau$ of $E$ onto a subfield $\bar{F}'$ such that $\tau(a) = \sigma(a)$ for all $a \in F$.

**Proof (continued).** If $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, consider $q(x) = \tau(a_0) + \tau(a_1)x + \tau(a_2)x^2 + \cdots + \tau(a_n)x^n \in K'[x] = \tau[K][x]$ where $\tau$ is from the maximal element $(K, \tau)$ of $S$. Since $\tau$ is an isomorphism, $q(x)$ is irreducible in $K'[x]$. Since $K' \leq \bar{F}'$, there is a zero $\alpha'$ of $q(x)$ in $\bar{F}'$. Similar to above, let $\psi_{\alpha'} : K'[x]/\langle q(x)\rangle \to K'(\alpha')$. Finally, let $\bar{\tau} : K[x]/\langle p(x)\rangle \to K'[x]/\langle q(x)\rangle$ be an isomorphism "extending" $\tau$ from $K$ to $K[x]/\langle p(x)\rangle$ (since $K \cong K'$ and $\tau$ "maps" $p(x) \in K[x]$ to $q(x) \in K'[x]$, then such an isomorphic relation exists). We have $\tau(x + \langle p(x)\rangle) = x + \langle q(x)\rangle$, for example.

# Theorem 49.3 (continued 9)

**Theorem 49.3.** Let $E$ be an algebraic extension of a field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$. Let $\bar{F}'$ be an algebraic closure of $F'$. Then $\sigma$ can be extended to an isomorphism $\tau$ of $E$ onto a subfield $\bar{F}'$ such that $\tau(a) = \sigma(a)$ for all $a \in F$.

**Proof (continued).** Then $\psi_{\alpha'}\bar{\tau}\psi_{\alpha}^{-1} : K(\alpha) \to K'(\alpha')$ is an isomorphism of $K(\alpha)$ onto a subfield $K'(\alpha')$ of $\bar{F}'$, since $K \leq K(\alpha)$ and $\psi_{\alpha'}\bar{\tau}\psi_{\alpha}^{-1}$ "extends" $\tau$ from $K$ (and so $\psi_{\alpha'}\bar{\tau}\psi_{\alpha}^{-1}(a) = \tau(a) = \sigma(a)$ for all $a \in F$). So $(K, \tau) < (K(\alpha), \psi_{\alpha'}\tau\psi_{\alpha}^{-1})$, which is a contradiction to the maximality of $(K, \tau)$. Therefore the assumption that $K \neq E$ is false and $K = E$.

# Theorem 49.3 (continued 9)

**Theorem 49.3.** Let $E$ be an algebraic extension of a field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$. Let $\bar{F}'$ be an algebraic closure of $F'$. Then $\sigma$ can be extended to an isomorphism $\tau$ of $E$ onto a subfield $\bar{F}'$ such that $\tau(a) = \sigma(a)$ for all $a \in F$.

**Proof (continued).** Then $\psi_{\alpha'}\bar{\tau}\psi_{\alpha}^{-1} : K(\alpha) \to K'(\alpha')$ is an isomorphism of $K(\alpha)$ onto a subfield $K'(\alpha')$ of $\bar{F}'$, since $K \leq K(\alpha)$ and $\psi_{\alpha'}\bar{\tau}\psi_{\alpha}^{-1}$ "extends" $\tau$ from $K$ (and so $\psi_{\alpha'}\bar{\tau}\psi_{\alpha}^{-1}(a) = \tau(a) = \sigma(a)$ for all $a \in F$). So $(K, \tau) < (K(\alpha), \psi_{\alpha'}\tau\psi_{\alpha}^{-1})$, which is a contradiction to the maximality of $(K, \tau)$. Therefore the assumption that $K \neq E$ is false and $K = E$. So $\tau$ is an isomorphism of $E = K$ onto a subfield $\tau(E) = \tau(K)$ of $\bar{F}'$ such that $\tau(a) = \sigma(a)$ on $F$, and the result follows. □

# Theorem 49.3 (continued 9)

**Theorem 49.3.** Let $E$ be an algebraic extension of a field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$. Let $\bar{F}'$ be an algebraic closure of $F'$. Then $\sigma$ can be extended to an isomorphism $\tau$ of $E$ onto a subfield $\bar{F}'$ such that $\tau(a) = \sigma(a)$ for all $a \in F$.

**Proof (continued).** Then $\psi_{\alpha'}\bar{\tau}\psi_\alpha^{-1} : K(\alpha) \to K'(\alpha')$ is an isomorphism of $K(\alpha)$ onto a subfield $K'(\alpha')$ of $\bar{F}'$, since $K \le K(\alpha)$ and $\psi_{\alpha'}\bar{\tau}\psi_\alpha^{-1}$ "extends" $\tau$ from $K$ (and so $\psi_{\alpha'}\bar{\tau}\psi_\alpha^{-1}(a) = \tau(a) = \sigma(a)$ for all $a \in F$). So $(K, \tau) < (K(\alpha), \psi_{\alpha'}\tau\psi_\alpha^{-1})$, which is a contradiction to the maximality of $(K, \tau)$. Therefore the assumption that $K \neq E$ is false and $K = E$. So $\tau$ is an isomorphism of $E = K$ onto a subfield $\tau(E) = \tau(K)$ of $\bar{F}'$ such that $\tau(a) = \sigma(a)$ on $F$, and the result follows. □

# Theorem 49.7

**Theorem 49.7.** Let $E$ be a finite extension of field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$, and let $\bar{F}'$ be an algebraic closure of $F'$. Then the number of extensions of $\sigma$ to an isomorphism $\tau$ of $E$ onto a subfield of $\bar{F}'$ is finite, and independent of $F'$, $\bar{F}'$, and $\sigma$. That is, the number of extensions is completely determined by the two fields $E$ and $F$.

**Proof.** Consider two isomorphisms $\sigma_1 : F \to F_1'$ and $\sigma_2 : F \to F_2'$ and let $\bar{F_1}'$ and $\bar{F_2}'$ be the algebraic closures of $F_1'$ and $F_2'$ respectively. Then $\sigma_2\sigma_1^{-1} : F_1' \to F_2'$ is an isomorphism.

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**Theorem 49.7.** Let $E$ be a finite extension of field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$, and let $\bar{F}'$ be an algebraic closure of $F'$. Then the number of extensions of $\sigma$ to an isomorphism $\tau$ of $E$ onto a subfield of $\bar{F}'$ is finite, and independent of $F'$, $\bar{F}'$, and $\sigma$. That is, the number of extensions is completely determined by the two fields $E$ and $F$.

**Proof.** Consider two isomorphisms $\sigma_1 : F \to F_1'$ and $\sigma_2 : F \to F_2'$ and let $\bar{F_1}'$ and $\bar{F_2}'$ be the algebraic closures of $F_1'$ and $F_2'$ respectively. Then $\sigma_2\sigma_1^{-1} : F_1' \to F_2'$ is an isomorphism. By Corollary 49.5, since $F_1' \cong F_2'$, then $\bar{F}_1' \cong \bar{F}_2'$ and by the Isomorphism Extension Theorem (Theorem 49.3) there is an isomorphism $\lambda : \bar{F}_1' \to \bar{F}_2'$ which extends the isomorphism $\sigma_2\sigma_1^{-1} : F_1' \to F_2'$.

# Theorem 49.7

**Theorem 49.7.** Let $E$ be a finite extension of field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$, and let $\bar{F}'$ be an algebraic closure of $F'$. Then the number of extensions of $\sigma$ to an isomorphism $\tau$ of $E$ onto a subfield of $\bar{F}'$ is finite, and independent of $F'$, $\bar{F}'$, and $\sigma$. That is, the number of extensions is completely determined by the two fields $E$ and $F$.

**Proof.** Consider two isomorphisms $\sigma_1 : F \to F_1'$ and $\sigma_2 : F \to F_2'$ and let $\bar{F_1}'$ and $\bar{F_2}'$ be the algebraic closures of $F_1'$ and $F_2'$ respectively. Then $\sigma_2\sigma_1^{-1} : F_1' \to F_2'$ is an isomorphism. By Corollary 49.5, since $F_1' \cong F_2'$, then $\bar{F}_1' \cong \bar{F}_2'$ and by the Isomorphism Extension Theorem (Theorem 49.3) there is an isomorphism $\lambda : \bar{F}_1' \to \bar{F}_2'$ which extends the isomorphism $\sigma_2\sigma_1^{-1} : F_1' \to F_2'$. Also by the Isomorphism Extension Theorem (Since $E$ is an extension field of $F$) there are isomorphisms $\tau_1$ extending $\sigma_1$ such that $\tau_1 : E \to \tau_1[E] \subset \bar{F_1}'$.

# Theorem 49.7

**Theorem 49.7.** Let $E$ be a finite extension of field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$, and let $\bar{F}'$ be an algebraic closure of $F'$. Then the number of extensions of $\sigma$ to an isomorphism $\tau$ of $E$ onto a subfield of $\bar{F}'$ is finite, and independent of $F'$, $\bar{F}'$, and $\sigma$. That is, the number of extensions is completely determined by the two fields $E$ and $F$.

**Proof.** Consider two isomorphisms $\sigma_1 : F \to F_1'$ and $\sigma_2 : F \to F_2'$ and let $\bar{F_1}'$ and $\bar{F_2}'$ be the algebraic closures of $F_1'$ and $F_2'$ respectively. Then $\sigma_2\sigma_1^{-1} : F_1' \to F_2'$ is an isomorphism. By Corollary 49.5, since $F_1' \cong F_2'$, then $\bar{F}_1' \cong \bar{F}_2'$ and by the Isomorphism Extension Theorem (Theorem 49.3) there is an isomorphism $\lambda : \bar{F}_1' \to \bar{F}_2'$ which extends the isomorphism $\sigma_2\sigma_1^{-1} : F_1' \to F_2'$. Also by the Isomorphism Extension Theorem (Since $E$ is an extension field of $F$) there are isomorphisms $\tau_1$ extending $\sigma_1$ such that $\tau_1 : E \to \tau_1[E] \subset \bar{F_1}'$.

# Theorem 49.7 (continued 1)

**Theorem 49.7.** Let $E$ be a finite extension of field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$, and let $\bar{F}'$ be an algebraic closure of $F'$. Then the number of extensions of $\sigma$ to an isomorphism $\tau$ of $E$ onto a subfield of $\bar{F}'$ is finite, and independent of $F'$, $\bar{F}'$, and $\sigma$. That is, the number of extensions is completely determined by the two fields $E$ and $F$.

**Proof (continued).** Now for each such $\tau_1$ we can define $\tau_2 = \lambda\tau_1$ where $\tau_2 : E \to \tau_2[E] \subset \bar{F}_2{}'$.
Since $\lambda$ extends $\sigma_2\sigma_1^{-1}$ and $\tau_1$ extends $\sigma_1$, then $\tau_2 = \lambda\tau_1$ extends $\sigma_2$.
Similarly, we could have defined $\tau_1$ in terms of $\tau_2$ as $\tau_1 = \lambda^{-1}\tau_2$.

# Theorem 49.7 (continued 1)

**Theorem 49.7.** Let $E$ be a finite extension of field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$, and let $\bar{F}'$ be an algebraic closure of $F'$. Then the number of extensions of $\sigma$ to an isomorphism $\tau$ of $E$ onto a subfield of $\bar{F}'$ is finite, and independent of $F'$, $\bar{F}'$, and $\sigma$. That is, the number of extensions is completely determined by the two fields $E$ and $F$.

**Proof (continued).** Now for each such $\tau_1$ we can define $\tau_2 = \lambda\tau_1$ where $\tau_2 : E \to \tau_2[E] \subset \bar{F_2}'$.
Since $\lambda$ extends $\sigma_2\sigma_1^{-1}$ and $\tau_1$ extends $\sigma_1$, then $\tau_2 = \lambda\tau_1$ extends $\sigma_2$. Similarly, we could have defined $\tau_1$ in terms of $\tau_2$ as $\tau_1 = \lambda^{-1}\tau_2$. So for each $\tau_1 : E \to \tau_1[E] \subseteq \bar{F_1}'$ there is a $\tau_2 : E \to \tau_2[E] \subseteq \bar{F_2}'$, and conversely.

# Theorem 49.7 (continued 1)

**Theorem 49.7.** Let $E$ be a finite extension of field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$, and let $\bar{F}'$ be an algebraic closure of $F'$. Then the number of extensions of $\sigma$ to an isomorphism $\tau$ of $E$ onto a subfield of $\bar{F}'$ is finite, and independent of $F'$, $\bar{F}'$, and $\sigma$. That is, the number of extensions is completely determined by the two fields $E$ and $F$.

**Proof (continued).** Now for each such $\tau_1$ we can define $\tau_2 = \lambda\tau_1$ where $\tau_2 : E \to \tau_2[E] \subset \bar{F_2}'$.
Since $\lambda$ extends $\sigma_2\sigma_1^{-1}$ and $\tau_1$ extends $\sigma_1$, then $\tau_2 = \lambda\tau_1$ extends $\sigma_2$. Similarly, we could have defined $\tau_1$ in terms of $\tau_2$ as $\tau_1 = \lambda^{-1}\tau_2$. So for each $\tau_1 : E \to \tau_1[E] \subseteq \bar{F_1}'$ there is a $\tau_2 : E \to \tau_2[E] \subseteq \bar{F_2}'$, and conversely. So there is a one to one correspondence between such $\tau_1$ and $\tau_2$, independent of $F'$ and $\bar{F}'$. So the number of extensions of $\sigma$ to an isomorphism $\tau$ of $E$ onto a subfield of $\bar{F}'$ is independent of $F'$, $\bar{F}'$, and $\sigma$.

# Theorem 49.7 (continued 1)

**Theorem 49.7.** Let $E$ be a finite extension of field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$, and let $\bar{F}'$ be an algebraic closure of $F'$. Then the number of extensions of $\sigma$ to an isomorphism $\tau$ of $E$ onto a subfield of $\bar{F}'$ is finite, and independent of $F'$, $\bar{F}'$, and $\sigma$. That is, the number of extensions is completely determined by the two fields $E$ and $F$.

**Proof (continued).** Now for each such $\tau_1$ we can define $\tau_2 = \lambda\tau_1$ where $\tau_2 : E \to \tau_2[E] \subset \bar{F}_2{}'$.
Since $\lambda$ extends $\sigma_2\sigma_1^{-1}$ and $\tau_1$ extends $\sigma_1$, then $\tau_2 = \lambda\tau_1$ extends $\sigma_2$.
Similarly, we could have defined $\tau_1$ in terms of $\tau_2$ as $\tau_1 = \lambda^{-1}\tau_2$. So for each $\tau_1 : E \to \tau_1[E] \subseteq \bar{F}_1{}'$ there is a $\tau_2 : E \to \tau_2[E] \subseteq \bar{F}_2{}'$, and conversely.
So there is a one to one correspondence between such $\tau_1$ and $\tau_2$, independent of $F'$ and $\bar{F}'$. So the number of extensions of $\sigma$ to an isomorphism $\tau$ of $E$ onto a subfield of $\bar{F}'$ is independent of $F'$, $\bar{F}'$, and $\sigma$.

# Theorem 49.7 (continued 2)

**Theorem 49.7.** Let $E$ be a finite extension of field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$, and let $\bar{F}'$ be an algebraic closure of $F'$. Then the number of extensions of $\sigma$ to an isomorphism $\tau$ of $E$ onto a subfield of $\bar{F}'$ is finite, and independent of $F'$, $\bar{F}'$, and $\sigma$. That is, the number of extensions is completely determined by the two fields $E$ and $F$.

**Proof (continued).** We now show that the number of extensions of $\sigma$ is finite. Since $E$ is a finite extension of $F$, then $E = F(\alpha_1, \alpha_2, ..., \alpha_n)$ for some $\alpha_1, \alpha_2, ..., \alpha_n$ in $E$, by Theorem 31.11.
For $\alpha_i \in \{\alpha_1, \alpha_2, ..., \alpha_n\}$, let $irr(\alpha_i, F) = a_{i0} + a_{i1}x + a_{i2}x^2 + ... + a_{im_i}x^{m_i}$ where $a_{ik} \in F$.

# Theorem 49.7 (continued 2)

**Theorem 49.7.** Let $E$ be a finite extension of field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$, and let $\bar{F}'$ be an algebraic closure of $F'$. Then the number of extensions of $\sigma$ to an isomorphism $\tau$ of $E$ onto a subfield of $\bar{F}'$ is finite, and independent of $F'$, $\bar{F}'$, and $\sigma$. That is, the number of extensions is completely determined by the two fields $E$ and $F$.

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For $\alpha_i \in \{\alpha_1, \alpha_2, ..., \alpha_n\}$, let $irr(\alpha_i, F) = a_{i0} + a_{i1}x + a_{i2}x^2 + ... + a_{im_i}x^{m_i}$ where $a_{ik} \in F$. Then
$a_{i0} + a_{i1}(\alpha_i) + a_{i2}(\alpha_i)^2 + ... + a_{im_i}(\alpha_i)^{m_i} = 0$ and so
$\tau(a_{i0} + a_{i1}(\alpha_i) + a_{i2}(\alpha_i)^2 + ... + a_{im_i}(\alpha_i)^{m_i}) = \tau(0)$ or
$\tau(a_{i0}) + \tau(a_{i1})\tau((\alpha_i)) + +... + \tau(a_{im_i})(\tau(\alpha_i))^{m_i} = 0$ or
$\sigma(a_{i0}) + \sigma(a_{i1})\tau((\alpha_i)) + ... + \sigma(a_{im_i})(\tau(\alpha_i))^{m_i} = 0$ since $\tau$ extends $\sigma$ and $a_{ik} \in F$.

# Theorem 49.7 (continued 2)

**Theorem 49.7.** Let $E$ be a finite extension of field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$, and let $\bar{F}'$ be an algebraic closure of $F'$. Then the number of extensions of $\sigma$ to an isomorphism $\tau$ of $E$ onto a subfield of $\bar{F}'$ is finite, and independent of $F'$, $\bar{F}'$, and $\sigma$. That is, the number of extensions is completely determined by the two fields $E$ and $F$.

**Proof (continued).** We now show that the number of extensions of $\sigma$ is finite. Since $E$ is a finite extension of $F$, then $E = F(\alpha_1, \alpha_2, \ldots, \alpha_n)$ for some $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $E$, by Theorem 31.11.
For $\alpha_i \in \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, let $irr(\alpha_i, F) = a_{i0} + a_{i1}x + a_{i2}x^2 + \ldots + a_{im_i}x^{m_i}$ where $a_{ik} \in F$. Then
$a_{i0} + a_{i1}(\alpha_i) + a_{i2}(\alpha_i)^2 + \ldots + a_{im_i}(\alpha_i)^{m_i} = 0$ and so
$\tau(a_{i0} + a_{i1}(\alpha_i) + a_{i2}(\alpha_i)^2 + \ldots + a_{im_i}(\alpha_i)^{m_i}) = \tau(0)$ or
$\tau(a_{i0}) + \tau(a_{i1})\tau((\alpha_i)) + + \ldots + \tau(a_{im_i})(\tau(\alpha_i))^{m_i} = 0$ or
$\sigma(a_{i0}) + \sigma(a_{i1})\tau((\alpha_i)) + \ldots + \sigma(a_{im_i})(\tau(\alpha_i))^{m_i} = 0$ since $\tau$ extends $\sigma$ and $a_{ik} \in F$.

# Theorem 49.7 (continued 3)

**Theorem 49.7.** Let $E$ be a finite extension of field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$, and let $\bar{F}'$ be an algebraic closure of $F'$. Then the number of extensions of $\sigma$ to an isomorphism $\tau$ of $E$ onto a subfield of $\bar{F}'$ is finite, and independent of $F'$, $\bar{F}'$, and $\sigma$. That is, the number of extensions is completely determined by the two fields $E$ and $F$.

**Proof (continued).** So the only possible value for $\tau(\alpha_i)$ is as a zero of $\sigma(a_{i0}) + \sigma(a_{i1})x + \sigma(a_{i2})x^2 + \ldots + \sigma(a_{im_i})x^{m_i} \in F[x]$ and hence there are only $m_i$ possible values for $\tau(\alpha_i)$. Therefore the number of values of $\tau$ on $\alpha_1, \alpha_2, \ldots, \alpha_n$ are finite and since a basis for $E = F(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is $1, \alpha_1, \alpha_2, \ldots, \alpha_n$, then there are only a finite number of extensions of $\sigma$ to $E$. □

# Theorem 49.7 (continued 3)

**Theorem 49.7.** Let $E$ be a finite extension of field $F$. Let $\sigma$ be an isomorphism of $F$ onto a field $F'$, and let $\bar{F}'$ be an algebraic closure of $F'$. Then the number of extensions of $\sigma$ to an isomorphism $\tau$ of $E$ onto a subfield of $\bar{F}'$ is finite, and independent of $F'$, $\bar{F}'$, and $\sigma$. That is, the number of extensions is completely determined by the two fields $E$ and $F$.

**Proof (continued).** So the only possible value for $\tau(\alpha_i)$ is as a zero of $\sigma(a_{i0}) + \sigma(a_{i1})x + \sigma(a_{i2})x^2 + ... + \sigma(a_{im_i})x^{m_i} \in F[x]$ and hence there are only $m_i$ possible values for $\tau(\alpha_i)$. Therefore the number of values of $\tau$ on $\alpha_1, \alpha_2, \ldots, \alpha_n$ are finite and since a basis for $E = F(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is $1, \alpha_1, \alpha_2, \ldots, \alpha_n$, then there are only a finite number of extensions of $\sigma$ to $E$. □

# Corollary 49.10

**Corollary 49.10.** If $F \leq E \leq K$ where $K$ is a finite extension field of the field $F$, then $\{K : F\} = \{K : E\}\{E : F\}$.

**Proof.** Let $\sigma : F \to \bar{F}$ (into) be an isomorphism of $E$ with $\sigma[E] \leq \bar{F}$ which fixes $F$. By definition, there are $\{E : F\}$ such $\sigma$. Let $\tau : K \to \bar{F}$ be an isomorphism of $K$ with $\tau[K] \leq \bar{F}$ which fixes $E$. By definition, there are $\{K : E\}$ such $\tau$.

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$$u(x) = \begin{cases} \sigma\tau(x) & \text{if} x \in E \\ \tau(x) & \text{if } x \in K \setminus E \end{cases} \tag{2}$$

is an isomorphism of $K$ with $u[K]$ which fixes $F$.

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So the number of such $u$ is $\{K : F\} = \{K : E\}\{E : F\}$ by the Multiplication Rule. □

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So the number of such $u$ is $\{K : F\} = \{K : E\}\{E : F\}$ by the Multiplication Rule. □