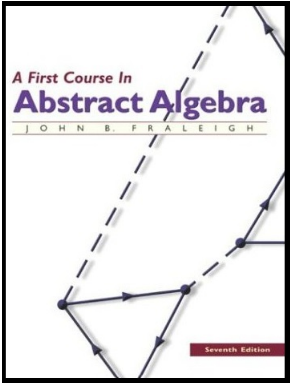
Introduction to Modern Algebra

Part X. Automorphisms and Galois Theory

X.50. Splitting Fields



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Theorem. 50.3. A field E, where $F \leq E \leq \bar{F}$, is a splitting field over F if and only if every automorphism of $\bar{F}$ leaving F fixed maps E onto itself (and this induces an automorphism of E leaving F fixed).

Proof. ($\Rightarrow$) Let E be a splitting field over F in $\bar{F}$ of $\{f_i(x) \mid i \in I\}$. Let σ be an automorphism of $\bar{F}$ leaving F fixed. Let $\{\alpha_j \mid j \in J\}$ be the set of all zeros in $\bar{F}$ of all the polynomials $f_i(x)$ for $i \in I$.

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$$g(\alpha_j) = a_0 + a_1\alpha_j + a_2\alpha_j^2 + \ldots + a_{n_j-1}\alpha_j^{n_j-1} \quad (1)$$

where n_j is the degree of $irr(\alpha_i, F)$ and each $a_k \in F$.

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where n_j is the degree of $irr(\alpha_i, F)$ and each $a_k \in F$. Consider the set S of all finite sums and finite products of elements of the form $g(\alpha_j)$ where $j \in J$. Then $S \subseteq E$ is closed under addition and multiplication, it contains 0,1, and is closed under the process of taking additive inverses (just replace the coefficients of $g(\alpha_j)$ with their additive inverses).

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Proof. (Continued) Since each element of S is in some finite extension of F, say $F(\alpha_{j_1}, \alpha_{j_2}, ..., \alpha_{j_r})$, then for any $s \in S$ $s \neq 0$, $s \in F(\alpha_{j_1}, \alpha_{j_2}, ..., \alpha_{j_r})$ we have $s^{-1} \in F(\alpha_{j_1}, \alpha_{j_2}, ..., \alpha_{j_r})$ and $s^{-1} \in S$. So S is a subfield of E and S contains all α_j for $j \in J$. Since E is the splitting of $\{f_i(x) \mid i \in I\}$ over F, then E is the smallest subfield of $\bar{F}$ containing F and all α_j for $j \in J$ (by definition of "splitting field"), so it must be that $S = F$ since S is a subfield of E containing all α_j for $j \in J$.

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Proof. (Continued) Since S consists of all finite sums of finite products of elements of the form $g(\alpha_j)$ for all $j \in J$ (where each g is a polynomial function with coefficients from F), then E also satisfies this — we ay that $\{\alpha_j \mid j \in J\}$ generates E over F (in the sense of taking finite sums and finite products, not in the sense of vector spaces discusses in the past). So for σ an automorphism of $\bar{F}$ which fixes F, the value of σ on E is determined by the value of $\sigma(\alpha_j)$ for $j \in J$. By Corollary 48.5, $\sigma(\alpha_j)$ must be the conjugate of α_j and so $\sigma(\alpha_j)$ must also be a zero of $irr(\alpha_j, F)$.

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Proof. (Continued) ($\Leftarrow$)

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Proof. If E is a splitting field over F in $\bar{F}$, then by Theorem 50.3a (the part for which we have given a proof), every automorphism of $\bar{F}$ induces an automorphism of E.

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Corollary 50.6. (Continued)

Corollary. 50.6. If $E \leq \bar{F}$ is a splitting field over F, then every irreducible polynomial in $F[x]$ having a zero in E splits in E.

Proof. (Continued) Since τ is defined on all of $\bar{F}$, then the range of τ^{-1} is all of $\bar{F}$. Hence, as in the proof of Theorem 50.3(a), $\tau[\tau^{-1}[\bar{F}]] = \bar{F}$ and so τ is an automorphism of $\bar{F}$ which fixes F. As commented earlier, by Theorem 50.3(a), τ induces an automorphism of E, and we have $\tau(\alpha) = \beta \in E$. Since β is an arbitrary zero of $f(x)$, then all zeroes of $f(x)$ are in E. That is, E splits $f(x)$. □

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Corollary 50.7.

Corollary. 50.7. If $E \leq \bar{F}$ is a splitting field over F, then every isomorphic mapping of E onto a subfield of $\bar{F}$ leaving F fixed is actually and automorphism of E. In particular, if E is a splitting field of finite degree over F, then $\{E : F\} = |G(E/F)|$, where $G(E/F)$ is the group of automorphisms of E leaving F fixed.

Proof. Every isomorphism σ mapping E onto a subfield of $\bar{F}$ leaving F fixed, can be extended to an isomrophism τ of $\bar{F}$ with a subfield of $\bar{F}$ by Theorem 49.3 (The Isomorphism Extension Theorem). By the argument in the proof of Corollary 50.6 (and considering τ^{-1}), we see that τ is onto $\bar{F}$ and so τ is an automorphism of $\bar{F}$.

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Proof. (Continued) Since $\{E : F\}$ is by definition, the number of different isomorphic mappings of E onto a subfield of $\bar{F}$ leaving F fixed, as shown above, such isomorphic mappings are all automorphisms of E (and of course an automorphism of E leaving F fixed is such a mapping). Since $G(E/F)$ is the group of automorphisms of E leaving F fixed, the result follows. □

Corollary 50.7. (Continued)

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