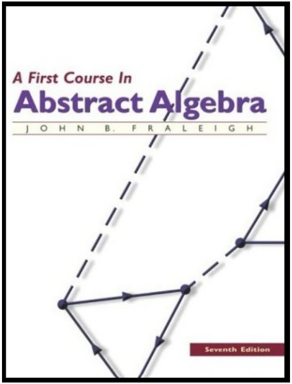
Introduction to Modern Algebra

Part I. Groups and Subgroups

I.3. Isomorphic Binary Structures



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Solution. Well, notice that φ is one-to-one and onto. However, consider $\varphi(1+2) = \varphi(3) = \varphi(3) + 1 = 4$ and $\varphi(1) + \varphi(2) = ((1)+1) + ((2)+1) = 5$ and so $\varphi(1+2) \neq \varphi(1) + \varphi(2)$ and φ and is NOT an isomorphism. □

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Theorem 3.14

Theorem 3.14 Suppose $\langle S, * \rangle$ has an identity element e. If $\varphi : S \to S'$ is an isomorphism of $\langle S, * \rangle$ with $\langle S', *' \rangle$, then $\varphi(e)$ is an identity element in $\langle S', *' \rangle$.

Proof. Let $s' \in S'$. Since φ is onto, then $\varphi(s) = s'$ for some $s \in S$. Then, since φ is an isomorphism, $\varphi(e) *' S' = \varphi(e) *' \varphi(s) = \varphi(e * s) = \varphi(s) = s'$. Similarly, $s' *' \varphi(e) = s'$. Therefore $\varphi(e)$ is an identity in $\langle S', *' \rangle$. □

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Exercise 3.16(a)

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Proof. For all $a, b \in \mathbb{Z}, \varphi(a + b) = a + b + 1$ and $\varphi(a) \circ \varphi(b) = (a + 1) \circ (b + 1) = a + b + 1$ and so $\varphi(a + b) = \varphi(a) \circ \varphi(b)$ and φ (being one-to-one and onto) is an isomorphism. Notice the identity in $\langle \mathbb{Z}, \circ \rangle$ is $\varphi(0) = 0 + 1 = 1$. □

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Proof. If $\varphi : S \to S'$ is one-to-one and onto, then $\varphi^{-1} : S' \to S$ is one-to-one and onto. Next, for all $a', b' \in S'$, there exists $a, b \in S$ such that $\varphi(a) = a'$ and $\varphi(b) = b'$. Also $\varphi^{-1}(a' *' b') = \varphi^{-1}(\varphi(a) *' \varphi(b)) = \varphi^{-1}(\varphi(a * b))$ (since φ is an isomorphism) $= a * b = \varphi^{-1}(a') * \varphi^{-1}(b')$. Therefore φ' is an isomorphism. □

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Exercise 3.33(b)

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Proof. For $z = a + ib \in \mathbb{C}$, define $\varphi(z) = \varphi(a + ib) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. Then φ is one-to-one and onto (right?). Also
$\varphi((a + ib) \cdot (c + id)) = \varphi((ac - bd) + i(ad + bc))$

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