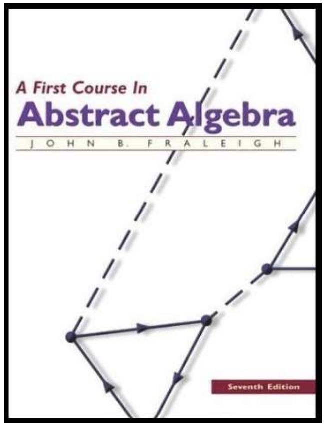
Introduction to Modern Algebra

Part I. Groups and Subgroups

I.7. Generating Sets and Cayley Digraphs



Theorem 7.4.

Theorem. The 7.4. The intersection of some subgroups H_i of a group G for $i \in I$ is again a subgroup of G. (**Note:** Set I is call an index set for the intersection. In general, the index set may not be finite – it may not even be countable.)

Proof. We use Theorem 5.14. First, for closure let $a, b \in \cap_{i \in I} H_i$. Then $a, b \in H_i$ for all $i \in I$. Since each H_i is a subgroup, $a, b \in H_i$ for all $i \in I$ and so $a, b \in \cap_{i \in I} H_i$. Next for the identity, $e \in H_i$ for all $i \in I$ since H_i is a group for all $i \in I$. Therefore $e \in \cap_{i \in I} H_i$. Finally, for $a \in \cap_{i \in I} H_i$ we have $a \in H_i$ for all $i \in I$. Since each H_i is a group, then $a^- \in H_i$ for all $i \in I$. So $a^{-1} \in \cap_{i \in I} H_i$. □

Theorem 7.6.

Theorem 7.6. If G is group and $a_i \in G$ for $i \in I$, then the subgroup H of G generated by $\{a_i \mid i \in I\}$ has as elements precisely those elements of G that are finite products of integral powers of the a_i, where the powers of a fixed a_i, may occur several times in the product.

Proof. Let K be the set of all products of integral powers the a_i. Since H is a group and $H \subset \{a_i \mid i \in I\}$, then $K \subset H$ (induction and the fact that H is closed; see Theorem 5.14).

Now we show that K is a subgroup containing $\{a_i \mid i \in I\}$, so $H \subset K$ and hence $K = H$. Notice that K is closed (products of elements of K are again in K). Since $(a_i)^0 = e$, $e \in K$. Let $k \in K$. Then

$$k = \left(a_{j_1}^{n_1}\right)\left(a_{j_2}^{n_2}\right)\cdots\left(a_{j_m}^{n_m}\right)$$

for some a_{i_ℓ} and n_ℓ, $\ell = 1, 2, \ldots, m$.

Theorem 7.6 (continued).

Theorem 7.6. If G is group and $a_i \in G$ for $i \in I$, then the subgroup H of G generated by $\{a_i \mid i \in I\}$ has as elements precisely those elements of G that are finite products of integral powers of the a_i, where the powers of a fixed a_i, may occur several times in the product.

Proof (Continued). Then

$$k^{-1} = \left(a_{j_1}^{-n_1}\right)\left(a_{j_2}^{-n_2}\right)\cdots\left(a_{j_m}^{-n_m}\right) \in K$$

. So, by Theorem 5.14, K is a subgroup of G containing $\{a_i \mid i \in I\}$ and (by the comment above), $H = K$. □