Part I. Groups and Subgroups
I.7. Generating Sets and Cayley Digraphs
Table of contents

1. Theorem 7.4.

2. Theorem 7.6.
Theorem. The intersection of some subgroups $H_i$ of a group $G$ for $i \in I$ is again a subgroup of $G$. (Note: Set $I$ is call an index set for the intersection. In general, the index set may not be finite – it may not even be countable.)

Proof. We use Theorem 5.14. First, for closure let $a, b \in \bigcap_{i \in I} H_i$. Then $a, b \in H_i$ for all $i \in I$. Since each $H_i$ is a subgroup, $a, b \in H_i$ for all $i \in I$ and so $a, b \in \bigcap_{i \in I} H_i$. 
Theorem 7.4. The intersection of some subgroups $H_i$ of a group $G$ for $i \in I$ is again a subgroup of $G$. (Note: Set $I$ is call an index set for the intersection. In general, the index set may not be finite – it may not even be countable.)

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□
Theorem 7.6.

**Theorem. 7.6.** If $G$ is group and $a_i \in G$ for $i \in I$, then the subgroup $H$ of $G$ generated by $\{a_i \mid i \in I\}$ has as elements precisely those elements of $G$ that are finite products of integral powers of the $a_i$, where the powers of a fixed $a_i$, may occur several times in the product.

**Proof.** Let $K$ be the set of all products of integral powers the $a_i$. Since $H$ is a group and $H \subset \{a_i \mid i \in I\}$, then $K \subset H$ (induction and the fact that $H$ is closed – Theorem 5.14). (Now we show that $K$ is a subgroup containing $\{a_i \mid i \in I\}$, so $H \subset K$ and hence $K = H$).
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$$k = (a_{j_1}^{n_1})(a_{j_2}^{n_2}) \cdots (a_{j_m}^{n_m})$$

for some $a_{i_\ell}$ and $n_\ell$, $\ell = 1, 2, \ldots, m$. 
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Theorem 7.6 (Continued). If $G$ is group and $a_i \in G$ for $i \in I$, then the subgroup $H$ of $G$ generated by $\{a_i \mid i \in I\}$ has as elements precisely those elements of $G$ that are finite products of integral powers of the $a_i$, where the powers of a fixed $a_i$, may occur several times in the product.

Proof (Continued). Then

$$k^{-1} = \left( a_{j_1}^{-n_1} \right) \left( a_{j_2}^{-n_2} \right) \cdots \left( a_{j_m}^{-n_m} \right) \in K$$

So, by Theorem 5.14, $K$ is a subgroup of $G$ containing $\{a_i \mid i \in I\}$ and (by the comment above), $H = K$.  

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