Introduction to Modern Algebra

Part I. Groups and Subgroups

I.7. Generating Sets and Cayley Digraphs

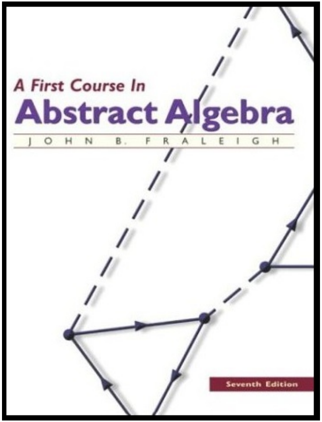

Table of contents

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Proof. We use Theorem 5.14. First, for closure let $a, b \in \cap_{i \in I} H_i$. Then $a, b \in H_i$ for all $i \in I$. Since each H_i is a subgroup, $a, b \in H_i$ for all $i \in I$ and so $a, b \in \cap_{i \in I} H_i$. Next for the identity, $e \in H_i$ for all $i \in I$ since H_i is a group for all $i \in I$. Therefore $e \in \cap_{i \in I} H_i$. Finally, for $a \in \cap_{i \in I} H_i$ we have $a \in H_i$ for all $i \in I$. Since each H_i is a group, then $a^{-} \in H_i$ for all $i \in I$. So $a^{-1} \in \cap_{i \in I} H_i$. □

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Proof. Let K be the set of all products of integral powers the a_i. Since H is a group and $H \subset \{a_i \mid i \in I\}$, then $K \subset H$ (induction and the fact that H is closed; see Theorem 5.14).

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Now we show that K is a subgroup containing $\{a_i \mid i \in I\}$, so $H \subset K$ and hence $K = H$. Notice that K is closed (products of elements of K are again in K). Since $(a_i)^0 = e$, $e \in K$. Let $k \in K$. Then

$$k = \left(a_{j_1}^{n_1}\right)\left(a_{j_2}^{n_2}\right)\cdots\left(a_{j_m}^{n_m}\right)$$

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Proof (Continued). Then

$$k^{-1} = \left(a_{j_1}^{-n_1}\right)\left(a_{j_2}^{-n_2}\right)\cdots\left(a_{j_m}^{-n_m}\right) \in K$$

. So, by Theorem 5.14, K is a subgroup of G containing $\{a_i \mid i \in I\}$ and (by the comment above), $H = K$. □