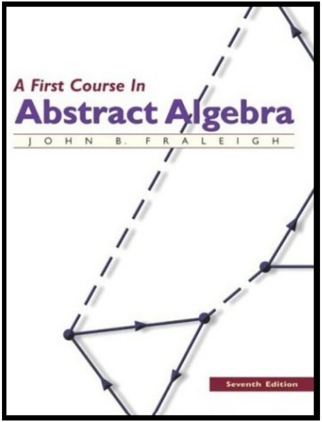
Introduction to Modern Algebra

Part Part II. Permutations, Cosets, and Direct Products

II.9. Orbits, Cycles, and the Alternating Groups



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Lemma

Lemma. Let σ be a permutation of set A. For $a, b \in A$, define $a \sim b$ if and only if $b = \sigma^n(A)$ for some $n \in \mathbb{Z}$. Then $\sim$ is an equivalence relation on A.

Proof. We need to establish that $\sim$ is reflexive, symmetric, and transitive.
First $a \sim a$ since $a = \iota(a) = \sigma^0(a)$.
Next, if $a \sim b$ then $b = \sigma^n(a)$ for some $n \in \mathbb{Z}$. Then $a = \sigma^{-n}(b)$ where $-n \in \mathbb{Z}$, and so $b \sim a$. So $\sim$ is symmetric.
Finally, if $a \sim b$ and $b \sim c$, then $b = \sigma^n(a)$ and $c = \sigma^m(b)$ for some $n, m \in \mathbb{Z}$. Then

$$c = \sigma^m(b) = \sigma^m(\sigma^n(n)) = \sigma^{m+n}(a)$$

and $c \sim a$. So $\sim$ is transitive. □

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Theorem 9.8.

Theorem. 9.8. Every permutation σ of a finite set is a product of disjoint cycles.

Proof. Let $B_1, B_2, \ldots, B_r$ be the disjoint orbits of σ. Let μ_i be the cycle defined by

$$\mu_i(x) = \begin{cases} \sigma(x) & \text{if } x \in B_i \\ x & \text{if } x \notin B_i \end{cases}.$$

So μ_i cycles around the elements of B_i while fixing the remaining elements of the finite set. Since the B_i's are disjoint, then the cycles (technically, the cyclic rotation of) μ_i are disjoint. By definition, $\sigma = \mu_1 \mu_2 \cdots \mu_r$. □

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Exercise 9.7.

Exercise. 9.7. Calculate in S_8 the product $(1\;4\;5)\,(7\;)\,(2\;5\;7)$. Remember to and from right to left!

Solution. The cyclic notation implies that 2 is mapped to 5 and then 5 is mapped to 1. Similarly we get:
$2 \to 5 \to 1,\; 5 \to 7 \to 8,\; 7 \to 2,\; 8 \to 7,\; 4 \to 5,\; 1 \to 4,\; 3 \to 3,\; 6 \to 6$ so the product in terms of disjoint cycles is $(2\;1\;4\;5\;8\;7)\,(3)\,(6)$. As a permutation this is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 1 & 3 & 5 & 8 & 6 & 2 & 7 \end{pmatrix}.$$

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$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix}..$$

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Lemma.

Lemma. For $n \geq 2$, the number of even permutations in S_n is $\frac{(n!)}{2}$.

Proof. Let A_n be the set of all even permutations in S_n and let B_n be the set of all odd permutations in S_n. Let τ be any given transposition in S_n. Define $\lambda_\tau : A_n \to B_n$ as $\lambda_\tau(\sigma) = \tau\sigma$ for all $\sigma \in A$. Since σ is an even permutation and τ is a transposition, then $\lambda_\tau(\tau) = \tau\sigma$ is an odd permutation and so $\lambda_\tau : A_n \to B_n$. Suppose for $\sigma, \mu \in A_n$ that $\lambda_\tau(\sigma) = \lambda_\tau(\mu)$. Then $\tau\sigma = \tau\mu$ and by left cancellation, $\sigma = \mu$. Therefore λ_τ is one-to-one.

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Theorem. 9.20. If $n \geq 2$, then the collection of all even permutations of $\{1, 2, 3, \ldots, n\}$ forms a subgroup of order $\frac{n!}{2}$ of the symmetry group S_n.

Proof. By Lemma, the set of all even permutations of S_n is a set A_n of size $\frac{n!}{2}$. Notice that the product of two even permutations is even, so A_n is closed under permutation multiplication. For $n \geq 2$, the identity ι in S_n is in A_n since $\iota = (1\,2)\,(2\,1)$. For any $\sigma \in A_n$, $\sigma = \tau_1 \tau_2 \cdots \tau_{2k}$ for some transpositions τ_i. Since a transpositions is its own inverse, $\sigma^{-1} = (\tau_1 \tau_2 \cdots \tau_{2k})^{-1} = \tau_{2k}^{-1} \cdots \tau_2^{-1} \tau_1^{-1} = \tau_{2k} \cdots \tau_2 \tau_1$ and $\sigma^{-1} \in A_n$. So, by Theorem 5.14, A_n is a subgroup of S_n. □

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