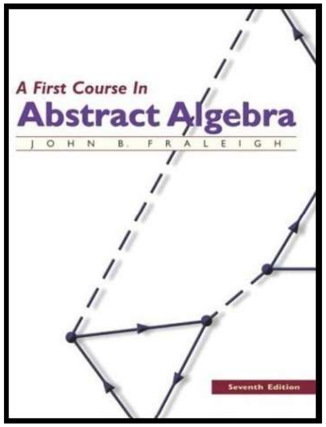
Introduction to Modern Algebra

Part III. Homomorphisms and Factor Groups

III.13. Homomorphisms



Example 13.2

Example 13.2. Suppose $\varphi : G \to G'$ is a homomorphism and φ is onto G'. If G is abelian then G' is abelian. Notice that this shows how we can get structure preservation without necessarily having an isomorphism.

Proof. Let $a', b' \in G'$. Since φ is onto G', there are $a, b \in G$ such that $\varphi(a) = a'$ and $\varphi(b) = b'$. Now

$$\begin{aligned} a'b' &= \varphi(a)\varphi(b) = \varphi(ab) \text{ since } \varphi \text{ is a homomorphism} \\ &= \varphi(ba) \text{ since } G \text{ is abelian} \\ &= \varphi(b)\varphi(a) \text{ since } \varphi \text{ is a homomorphism} \\ &= b'a' \end{aligned}$$

So G' is abelian, as claimed. □

Example 13.8

Example 13.8. Let $G = G_1 \times G_2 \times \cdots \times G_i \times \cdots \times G_n$ be a direct product of groups $G_1, G_2, \ldots, G_n$. Define the *projection map* $\pi_i : G \to G_i$ where $\pi_i((g_1, g_2, \ldots, g_i, \ldots, g_n)) = g_i$. Then π_i is a homomorphism.

Proof. Let $(g_1, g_2, \ldots, g_i, \ldots, g_n), (h_1, h_2, \ldots, h_i, \ldots, h_n) \in G$. Then

$$\pi_i((g_1, g_2, \ldots, g_i, \ldots, g_n)(h_1, h_2, \ldots, h_i, \ldots, h_n))$$

$$= \pi_i((g_1h_1, g_2h_2, \ldots, g_ih_i, \ldots, g_nh_n)) \text{ using multiplication notation}$$

$$= g_ih_i = \pi_i((g_1, g_2, \ldots, g_i, \ldots, g_n))\pi_i((h_1, h_2, \ldots, h_i, \ldots, h_n)).$$

So π_i is a homomorphism, as claimed. □

Exercise 13.10

Exercise 13.10. Let F be the *additive* group of all continuous functions mapping $\mathbb{R}$ into $\mathbb{R}$. Let $\mathbb{R}$ be the *additive* group of real numbers and let $\varphi : F \to \mathbb{R}$ be given by $\varphi(f) = \int_0^4 f(x)\,dx$. Then φ is a homomorphism.

Proof. Let $f_1, f_2 \in F$. Then

$$\varphi(f_1 + f_2) = \int_0^4 (f_1(x) + f_2(x))\,dx$$

$$= \int_0^4 f_1(x)\,dx + \int_0^4 f_2(x)\,dx = \varphi(f_1) + \varphi(f_2).$$

So φ is a homomorphism. □

Theorem 13.12

Theorem 13.12. Let φ be a homomorphism of a group G into group G'.

(1) If e is the identity in G, then $\varphi(e)$ is the identity element e' in G'.
(2) If $a \in G$ then $\varphi(a^{-1}) = (\varphi(a)^{)}-1$.
(3) If H is a subgroup of G, then $\varphi[H]$ is a subgroup of G'.
(4) If K' is a subgroup of G', then $\varphi^{-1}[K]$ is a subgroup of G.

Proof. Let φ be a homomorphism of G into G'.

(1) We have $\varphi(a) = \varphi(ae) = \varphi(a)\varphi(e)$ and so $e' = (\varphi(a))^{-1}(\varphi(a)) = \varphi(a)^{-1}(\varphi(a)\varphi(e))$ or $e' = \varphi(e)$, as claimed.

(2) We have $e' = \varphi(e) = \varphi(aa^{-1}) = \varphi(a)\varphi(a^{-1})$ and so $(\varphi(a))^{-1} = \varphi(a^{-1})$, as claimed.

Theorem 13.12 (continued 1)

Theorem 13.12. Let φ be a homomorphism of a group G into group G'.

(1) If e is the identity in G, then $\varphi(e)$ is the identity element e' in G'.
(2) If $a \in G$ then $\varphi(a^{-1}) = (\varphi(a)^{)}-1$.
(3) If H is a subgroup of G, then $\varphi[H]$ is a subgroup of G'.
(4) If K' is a subgroup of G', then $\varphi^{-1}[K]$ is a subgroup of G.

Proof (continued). Let φ be a homomorphism of G into G'.

(3) Let $H < G$ and consider $\varphi[H]$. For any $\varphi(a), \varphi(b) \in \varphi[H]$ we have $\varphi(a)\varphi(b) = \varphi(ab) \in \varphi[H]$ (since $a, b \in H$ implies $ab \in H$). By (1) $e' = \varphi(e) \in \varphi[H]$ (since $e \in H$) and by (2) for all $\varphi(a) \in \varphi[H]$, $\varphi(a)^{-1} = \varphi(a^{-1}) = \varphi(a^{-1}) \in \varphi[H]$ (since $a \in H$ implies $a^{-1} \in H$). So $\varphi[H]$ is a subgroup of G' by Theorem 5.14, as claimed.

Theorem 13.12 (continued 2).

Theorem 13.12. Let φ be a homomorphism of a group G into group G'.

(1) If e is the identity in G, then $\varphi(e)$ is the identity element e' in G'.
(2) If $a \in G$ then $\varphi(a^{-1}) = (\varphi(a)^{)}-1$.
(3) If H is a subgroup of G, then $\varphi[H]$ is a subgroup of G'.
(4) If K' is a subgroup of G', then $\varphi^{-1}[K]$ is a subgroup of G.

Proof (continued). Let φ be a homomorphism of G into G'.

(4) Let $K' < G'$ and consider $\varphi^{-1}[K]$. For any $a, b \in \varphi^{-1}[K]$ we have $\varphi(a), \varphi(b) \in K$ and so $\varphi(a)\varphi(b) \in K$; therefore $ab \in \varphi^{-1}[K]$. Since $e' \in K$ (K is a subgroup), then $\varphi(e) = e' \in K$ (by (1)) implies $e \in \varphi^{-1}[K]$. For all $a \in \varphi^{-1}[K]$ we have $\varphi(a)^{-1} \in K$ and since $\varphi(a)^{-1} = \varphi(a^{-1})$, by (2) $\varphi(a^{-1}) \in K$ and $a^{-1} \in \varphi^{-1}[K]$. So $\varphi^{-1}[K]$ is a subgroup of G by Theorem 5.14, as claimed. □

Exercise 13.18

Exercise 13.18. Let $\varphi : \mathbb{Z} \to \mathbb{Z}_{10}$ be a homomorphism such that $\varphi(1) = 6$. Find $\text{Ker}(\varphi)$.

Solution. Since $\langle 1 \rangle = \mathbb{Z}$, we can find all values of φ on $\mathbb{Z}$. The identity of $\mathbb{Z}_{10}$ is 0, so any element of $\mathbb{Z}$ mapped to $0(\text{mod } 10)$ is in the kernel of φ. Now for $j \in \mathbb{Z}$ we have

$$\varphi(j) = \varphi(j \cdot 1) = \underbrace{\varphi(1) + \varphi(1) + \cdots + \varphi(1)}_{j \text{ times}} = j\varphi(1) = 6j.$$

So $\varphi(j) \equiv 0(\text{mod } 10)$ implies $6j \equiv 0(\text{mod } 10)$ or $6j = 10k$ for some k, or $3j = 5k$ for some k. We see that j must be a multiple of 5, or $j \equiv 0(\text{mod } 5)$. So $\text{Ker}(\varphi) = \{j \in \mathbb{Z} \mid j \equiv 0(\text{mod } 5)\}$. □

Theorem 13.15

Theorem 13.15. Let $\varphi : G \to G'$ be a group homomorphism and let $H = \text{Ker}(\varphi)$. Let $a \in G$. Then the set $\varphi^{-1}[\{\varphi(a)\}] = \{x \in G \mid \varphi(x) = \varphi(a)\}$ is the left coset aH of H, and is also the right coset Ha of H.

Proof. We need to show that $\varphi^{-1}[\varphi(a)] = \{x \in G \mid \varphi(x) = \varphi(a)\} = aH = Ha$ for each $a \in G$, where $H = \text{Ker}(\varphi)$. First we show this for the left coset aH. Suppose φ maps both a and x to the same element of G' (we must to show that a and x are in the same coset, namely aH). Then $\varphi(a) = \varphi(x)$ or

$$\varphi(a)^{-1}\varphi(x) = e' \qquad (*)$$

or $\varphi(a^{-1})\varphi(x) = e'$ by Theorem 13.12(2), or $\varphi(a^{-1}x) = e'$ since φ is a homomorphism. So $a^{-1}x \in \text{Ker}(\varphi) = H$ (by hypothesis) and so $a^{-1}x = h$ for some $h \in H$. That is $x = ah \in ah$. So if a and x are mapped by φ to the same element of G', then $a, x \in aH$. Therefore $\{x \in G \mid \varphi(x) = \varphi(a)\} \subseteq aH$.

Theorem 13.15 (continued).

Theorem 13.15. Let $\varphi : G \to G'$ be a group homomorphism and let $H = \text{Ker}(\varphi)$. Let $a \in G$. Then the set $\varphi^{-1}[\{\varphi(a)\}] = \{x \in G \mid \varphi(x) = \varphi(a)\}$ is the left coset aH of H, and is also the right coset Ha of H.

Proof (continued). Next, let

$$y \in aH, \qquad (**)$$

or $y = ah$ for some $h \in H$. Then $\varphi(y) = \varphi(ah) = \varphi(a)\varphi(h) = \varphi(a)e' = \varphi(a)$ ($\varphi(h) = e'$ since $H = \text{Ker}(e)$). So $y \in \{x \in G \mid \varphi(x) = \varphi(a)\}$ and $aH \subset \{x \in G \mid \varphi(x) = \varphi(a)\}$. Therefore $aH = \{x \in G \mid \varphi(x) = \varphi(a)\}$. To show the results for coset Ha, simply replace $(*)$ with $\varphi(x)\varphi(a)^{-1} = e'$ and $(**)$ with $y \in Ha$ (this is Exercise 13.52). □

Corollary 13.18

Corollary 13.18. A group homomorphism $\varphi : G \to G'$ is a one-to-one map if and only if $\text{Ker}(\varphi) = \{e\}$.

Proof. If $\text{Ker}(\varphi) = \{e\}$ then for $a, b \in G$, $a \neq b$, we have by Theorem 13.15 that $\varphi(a) = a\text{Ker}(\varphi) = \{a\}$ and $\varphi(b) = b\text{Ker}(\varphi) = \{b\}$, and so $\varphi(a) \neq \varphi(b)$. That is, φ is one to one, as claimed.

Suppose φ is one to one. By Theorem 13.12(i), we have $\varphi(e) = e'$ and if φ is one-to-one then no other element of G is mapped to e'. That is, $\text{Ker}(\varphi) = \{e\}$, as claimed. □

Exercise 13.34

Exercise 13.34. Is there a nontrivial homomorphism from $\mathbb{Z}_{12}$ to $\mathbb{Z}_4$?

Solution. We take our lead from the picture for Theorem 13.15. We need to swap the cosets of $\text{Ker}(\varphi)$ into $\mathbb{Z}_4$. One way to do this is to partition $\mathbb{Z}_{12}$ into four cosets each of size three (another way is to use two cosets each of size six; or six cosets each of size two). This be achieved as follows:

	0	1	2	3
$\mathbb{Z}_{12}$	4	5	6	7
	8	9	10	11
φ	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathbb{Z}_4$	0	1	2	3

Then $\varphi(x) = x(\text{mod } 4)$, $\text{Ker}(\varphi) = \{0, 4, 8\}$, and the cosets are $1\,\text{Ker}(\varphi) = \{1, 5, 9\}$, $2\,\text{Ker}(\varphi) = \{2, 6, 10\}$, and $3\,\text{Ker}(\varphi) = \{3, 7, 11\}$. □

Corollary 13.20

Corollary 13.20. If $\varphi : G \to G'$ is a homomorphism, then Ker (φ) is a normal subgroup of G.

Proof. We know by Theorem 13.15 that for $H = \text{Ker}(\varphi)$, left cosets and right cosets coincide. That is $aH = Ha$ for all $a \in G$. So, by definition, $H = \text{Ker}(\varphi)$ is a normal subgroup. □

Exercise 13.50

Exercise 13.50. Let $\varphi : G \to H$ be a group homomorphism. Then $\varphi[G]$ is abelian if and only if for all $x, y \in G$ we have $xyx^{-1}y^{-1} \in \text{Ker}(\varphi)$.

Proof. Let $x', y' \in \varphi[G]$. Then $x' = \varphi(x)$ and $y' = \varphi(y)$ for some $x, y \in G$. Now assuming $xyx^{-1}y^{-1} \in \text{Ker}(\varphi)$,

$$\begin{aligned} y'x' &= \varphi(y)\varphi(x) \\ &= e'\varphi(y)\varphi(x) \\ &= \varphi(xyx^{-1}y^{-1})\varphi(y)\varphi(x) \\ &= \varphi(xyx^{-1}y^{-1}yx) \text{ since } \varphi \text{ is a homomorphism} \\ &= \varphi(xy) \\ &= \varphi(x)\varphi(y) \\ &= x'y'. \end{aligned}$$

So, $\varphi[G]$ is abelian, as claimed.

Exercise 13.50 (continued)

Exercise 13.50. Let $\varphi : G \to H$ be a group homomorphism. Then $\varphi[G]$ is abelian if and only if for all $x, y \in G$ we have $xyx^{-1}y^{-1} \in \text{Ker}(\varphi)$.

Proof (continued). Suppose $\varphi[G]$ is abelian. Let $x, y \in G$. Then

$$\begin{aligned} \varphi(xyx^{-1}y^{-1}) &= \varphi(x)\varphi(y)\varphi(x^{-1})\varphi(y^{-1}) \text{ since } \varphi \\ &\quad \text{is a homomorphism} \\ &= \varphi(x)\varphi(x^{-1})\varphi(y)\varphi(y^{-1}) \\ &\quad \text{since } \varphi[G] \text{ is abelian} \\ &= \varphi(x)\varphi(x^{-1})\varphi(y)\varphi(y^{-1}) \\ &\quad \text{by Theorem 13.12 part (2)} \\ &= e'. \end{aligned}$$

So $xyx^{-1}y^{-1} \in \text{Ker}(\varphi)$, as claimed. □