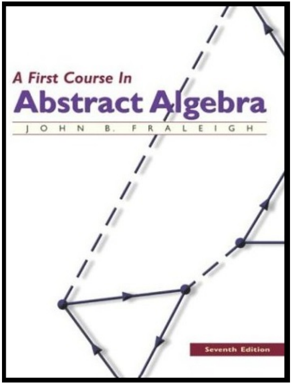
Introduction to Modern Algebra

Part IV. Rings and Fields

IV.21. The Field of Quotients of an Integral Domain



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Thirdly, suppose $(a, b) \sim (c, d)$ and $(c, d) \sim (r, s)$. Then $ad = bc$ and $cs = dr$. Therefore

$$\begin{aligned} asd &= sad \text{ by commutativity} \\ &= sbc \text{ since } ad = bc \\ &= bcs \text{ by commutativity} \\ &= bdr \text{ since } cs = dr \\ &= brd \text{ by commutativity.} \end{aligned}$$

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Lemma 21.2 (continued).

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Proof (continued). So $(a, b) \sim (c, d)$ and $(c, d) \sim (r, s)$ implies $asd = brd$. Since $d \neq 0$ and D is an integral domain (no divisors of 0), then by Theorem 19.5 the laws of cancellation hold and so $as = br$. That is $(a, b) \sim (r, s)$ and $\sim$ is transitive.

Therefore $\sim$ is an equivalence relation, as claimed. $\square$

Lemma 21.3.

Lemma 21.3. For $[(a, b)], [(c, d)] \in F$, the equations

$$[(a, b)] + [(c, d)] = [(ad + bc, bd)] \text{ and } [(a, b)] \cdot [(c, d)] = [(ac, bd)]$$

give well-defined operations of addition and multiplication on F.

Proof. First, notice for $[(a, b)], [(c, d)] \in F$, we have $(a, b), (c, d) \in S$ with $b \neq 0$ and $d \neq 0$. So $bd \neq 0$ since 0 is an integral domain and $(ad + bc, bd), (ac, bd) \in S$ and so the right hand sides of the two equations are in fact elements of F.

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Now to show the independence of the choice of representatives from the equivalence classes. Let $(a_1, b_1) \in [(a,b))]$ and $(c_1, d_1) \in [(c,d)]$. Then $(a_1, b_1) \sim (a,b)$ and $(c_1, d_1) \sim (c,d)$. So $a_1 b = b_1 a$ and $c_1 d = d_1 c$. So $a_1 b (d_1 d) + c_1 d (b_1 b) = b_1 a (d_1 d) + d_1 c (b_1 b)$ and so $(a_1 d_1 + b_1 c_1) bd = b_1 d_1 (ad + bc)$ and then $(a_1 d_1 + b_1 c_1, b_1 d_1) \sim (ad + bc, bd)$.

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give well-defined operations of addition and multiplication on F.

Proof (continued). That is $(a_1d_1 + b_1c_1, b_1d_1) \in [(ad+bc, bd)]$. So addition in F is well defined. As above, $a_1b = b_1a$ and $c_1d = d_1c$ imply $a_1b(c_1d) = b_1a(d_1c)$ or $a_1c_1bd = b_1d_1ac$. That is $(a_1c_1, b_1d_1) \sim (ac, bd)$, and $(a_1c_1, b_1d_1) \sim [(ac, bd)]$. So multiplication is F is well defined, as claimed. □

Lemma 21.3 (continued).

Lemma 21.3. For $[(a, b)], [(c, d)] \in F$, the equations

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Proof (continued). That is $(a_1 d_1 + b_1 c_1, b_1 d_1) \in [(ad + bc, bd)]$. So addition in F is well defined. As above, $a_1 b = b_1 a$ and $c_1 d = d_1 c$ imply $a_1 b (c_1 d) = b_1 a (d_1 c)$ or $a_1 c_1 bd = b_1 d_1 ac$. That is $(a_1 c_1, b_1 d_1) \sim (ac, bd)$, and $(a_1 c_1, b_1 d_1) \sim [(ac, bd)]$. So multiplication is F is well defined, as claimed. □

Lemma 21.A.

Lemma 21.A. F as defined above is a field. That is,

1. $+$ in F is commutative.
2. $+$ in F is associative.
3. $[(0,1)]$ is the additive identity in F.
4. $[(-a,b)]$ is the additive inverse for $[(a,b)]$ in F.
5. $\cdot$ is associative in F.
6. $\cdot$ is commutative in F.
7. The distribution laws hold in F:
   $[(a,b)]\cdot([(c,d)]+[(r,s)]) = [(a,b)]\cdot[(c,d)]+[(a,b)]\cdot[(r,s)]$
   (right distribution will follow from commutativity of $\cdot$).
8. $[(1,1)]$ is the multiplicative identity in F.
9. If $[(a,b)] \in F$, $[(a,b)] \neq [(0,1)]$, then $[(b,a)] \in F$ is the multiplicative inverse of $[(a,b)]$.

Lemma 21.A (continued 1).

Lemma 21.A.

1. $+$ in F is commutative.
2. $+$ in F is associative.

Proof.

1. Let $[(a, b)] \in F$. Then
$[(a, b)] + [(c, d)] = [(ad + bc, bd)] = [(cb + da, db)]$ since
$ad + bc = cb + da$ and $bd = db$ in integral domain D.

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$ad + bc = cb + da$ and $bd = db$ in integral domain D.

2. Let $[(a, b)], [(c, d)], [(r, s)] \in F$. Then
$([(a, b)] + [(c, d)]) + [(r, s)] = [(ad + bc, bd)] + [(r, s)] =$
$[(ad + bc)\, s + (bd)\, r, (bd)\, s] = [a\,(ds) + b\,(cs + dr)\,, b\,(ds)] =$
$[(a, b)] + [(cs + dr, ds)] = [(a, b)] + ([(c, d)] + [(r, s)])$ and $+$ is associative.

Lemma 21.A (continued 1).

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$([(a, b)] + [(c, d)]) + [(r, s)] = [(ad + bc, bd)] + [(r, s)] =$
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Lemma 21.A (continued 2).

Lemma 21.A.

3. $[(0,1)]$ is the additive identity in F.
4. $[(-a,b)]$ is the additive inverse for $[(a,b)]$ in F.
5. $\cdot$ is associative in F.

Proof.
3. Let $[(a,b)] \in F$, then
$[(a,b)] + [(0,1)] = [(a(1) + b(0), b(1))] = [(a,b)]$. Since $+$ is commutative, $[(0,1)] + [(a,b)] = [(a,b)]$ and $[(0,1)]$ is the additive identity.

4. For $[(a,b)] \in F$, $b \neq 0$ and so $[(-a,b)] \in F$. Now $[(a,b)] + [(-a,b)] = [a(b) + b(-a), b] = [0,b]$. Now $[0,1] \sim [0,b]$ since $0 \cdot b = 1 \cdot 0$. So $[(a,b)] + [(-a,b)] = [(0,1)]$ and since $+$ is commutative, $[(-a,b)] + [(a,b)] = [(0,1)]$ and $[(-a,b)]$ is the $+$ inverse of $[(a,b)]$.

Lemma 21.A (continued 2).

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4. For $[(a,b)] \in F$, $b \neq 0$ and so $[(-a,b)] \in F$. Now $[(a,b)]+[(-a,b)]=[a(b)+b(-a),b]=[0,b]$. Now $[0,1] \sim [0,b]$ since $0 \cdot b = 1 \cdot 0$. So $[(a,b)]+[(-a,b)]=[(0,1)]$ and since $+$ is commutative, $[(-a,b)]+[(a,b)]=[(0,1)]$ and $[(-a,b)]$ is the $+$ inverse of $[(a,b)]$.

Lemma 21.A (continued 3).

Lemma 21.A.

5. $\cdot$ is associative in F.
6. $\cdot$ is commutative in F.

Proof (continued).
5. Let $[(a,b)], [(c,d)], [(r,s)] \in F$. Then
$[(a,b)] \cdot ([(c,d)] \cdot [(r,s)]) = [(a,b)] \cdot [(cr,ds)] = [(acr,bds)] =$
$[(ac,bd)] \cdot [(r,s)] = ([(a,b)] \cdot [(c,d)]) \cdot [(r,s)]$ and $\cdot$ is associative.

6. Let $[(a,b)], [(c,d)] \in F$. Then
$[(a,b)] \cdot [(c,d)] = [(ac,bd)] = [(ca,db)]$, since $\cdot$ is commutative in D,
$= [(c,d)] \cdot [(a,b)]$. So $\cdot$ is commutative in F.

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Lemma 21.A (continued 4).

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7. The distribution laws hold in F:
$[(a,b)]\cdot([(c,d)]+[(r,s)])=[(a,b)]\cdot[(c,d)]+[(a,b)]\cdot[(r,s)]$
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Proof (continued).
7. We have $[(a,b)]\cdot([(c,d)]+[(r,s)])=[(a,b)]\,[(cs+dr,ds)]=$
$[(a\,(cs+dr)\,,b\,(ds))]=[(acs+adr,bds)]=[(acs,bds)]+[(adr,bds)]=$
$[(ac,bd)]+[(ar,bs)]$, since $(acs,bds)\sim(ac,bd)$ and
$(adr,bds)\sim(ar,bs)$, Therefore
$[(a,b)]\cdot([(c,d)]+[(r,s)])=[(a,b)]\cdot[(c,d)]+[(a,b)]\cdot[(r,s)]$, as claimed.

8. Let $[(a,b)]\in F$. Then $[(a,b)]\cdot[(1,1)]=[(a\,(1)\,,b\,(1))]=[(a,b)]$.
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9. If $[(a,b)] \in F$, $[(a,b)] \neq [(0,1)]$, then $[(b,a)] \in F$ is the multiplicative inverse of $[(a,b)]$.

Proof (continued).
9. Since $[(a,b)] \neq [(0,1)]$, then $a \neq 0$ and so $[(b,a)] \in F$. Now $[(a,b)] \cdot [(b,a)] = [(ab,ba)]$. Since $(ab,ba) = (ab,ab) \sim (1,1)$, then $[(a,b)] \cdot [(b,a)] = [(1,1)]$. Since $\cdot$ is commutative by (6), $[(b,a)] \cdot [(a,b)] = [(1,1)]$ and $[(b,a)]$ is the multiplicative inverse of $[(a,b)]$. $\square$

Lemma 21.4.

Lemma 21.4. The map $i : D \rightarrow F$ given by $i(a) = [(a, 1)]$ is an isomorphism of D with a subring of F.

Proof. First,

$$i(a+b) = [(a+b,1)] = [(a,1)+(b,1)] = [(a,1)]+[(b,1)] = i(a)+i(b).$$

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Now to show i is one-to-one. Suppose $i(a) = i(b)$ then $[(a, 1)] = [(b, 1)]$ and so $(a, 1) \sim (b, 1)$, or $a1 = 1b$, or $a = b$. So i is one-to-one and i preserves sums and products.

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Theorem 21.6.

Theorem 21.6. Let F be a field of quotients of D and let L be any field containing D. Then there exists a map $\psi : F \to L$ that gives an isomorphism of F with a subfield of L such that $\psi(a) = a$ for $a \in D$.

Proof. By definition, any element of F is a quotient of elements in $i[D]$. If $f \in F$ satisfies $f = i(a) \cdot (i(b))^{-1}$, then we denote this as $f = a/_F b$ (here, 'a' and 'b' are treated as elements of F^* although they are not, but their images $i(a)$ and $i(b)$ are in F). Define $\psi : F \to L$ as $\psi(a) = a$ for $a \in D$ $\psi(f) = \psi(a)/_L\psi(b)$ for $f = a/_F b \in F \backslash i[D]$.

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Theorem 21.6 (continued 1).

Theorem 21.6. Let F be a field of quotients of D and let L be any field containing D. Then there exists a map $\psi : F \to L$ that gives an isomorphism of F with a subfield of L such that $\psi(a) = a$ for $a \in D$.

Proof (continued). Now if $f = a/_F b = c/_F d$, then $ad = bc$ (as a product of elements of D) and so $\psi(ad) = \psi(bc)$. But since ψ is the identity on D, $\psi(ad) = \psi(a)\psi(d)$ and $\psi(bc) = \psi(b)\psi(c)$. So $\psi(a)\psi(d) = \psi(b)\psi(c)$ and $\psi(a)/_L\psi(b) = \psi(c)/_L\psi(d)$. Therefore $\psi(f)$ is well-defined.

Now to show that ψ is an isomorphism. First, let $x, y \in F$, so $x = a/_F b$ and $y = c/_F d$ for some $a, b, c, d \in D$, $b \neq 0$, $d \neq 0$. Then $\psi(xy) = \psi((a/_F b) \cdot (c/_F d)) = \psi((ac)/_F(bd)) = \psi(ac)/_L\psi(bd)$, by definiton of ψ on F, $= (ac)/_L(bd)$, since ψ is identity on D, $= (ac)/_L(bd)$, since ψ is identity on D, $= (a/_L b)(c/_L d)$, since D is integral domain, $= \psi(a/_F b)\psi(c/_F d) = \psi(x)\psi(y)$.

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Theorem 21.6. Let F be a field of quotients of D and let L be any field containing D. Then there exists a map $\psi : F \to L$ that gives an isomorphism of F with a subfield of L such that $\psi(a) = a$ for $a \in D$.

Proof (continued). Now if $f = a/_F b = c/_F d$, then $ad = bc$ (as a product of elements of D) and so $\psi(ad) = \psi(bc)$. But since ψ is the identity on D, $\psi(ad) = \psi(a)\psi(d)$ and $\psi(bc) = \psi(b)\psi(c)$. So $\psi(a)\psi(d) = \psi(b)\psi(c)$ and $\psi(a)/_L\psi(b) = \psi(c)/_L\psi(d)$. Therefore $\psi(f)$ is well-defined.

Now to show that ψ is an isomorphism. First, let $x, y \in F$, so $x = a/_F b$ and $y = c/_F d$ for some $a, b, c, d \in D$, $b \neq 0$, $d \neq 0$. Then $\psi(xy) = \psi((a/_F b) \cdot (c/_F d)) = \psi((ac)/_F(bd)) = \psi(ac)/_L\psi(bd)$, by definiton of ψ on F, $= (ac)/_L(bd)$, since ψ is identity on D, $= (ac)/_L(bd)$, since ψ is identity on D, $= (a/_L b)(c/_L d)$, since D is integral domain, $= \psi(a/_F b)\psi(c/_F d) = \psi(x)\psi(y)$.

Theorem 21.6 (continued 2).

Theorem 21.6. Let F be a field of quotients of D and let L be any field containing D. Then there exists a map $\psi : F \to L$ that gives an isomorphism of F with a subfield of L such that $\psi(a) = a$ for $a \in D$.

Proof (continued). Next $\psi(x+y) = \psi(a/_F b + c/_F d) =$
$\psi((ad+bc)/_F(bd)) = \psi(ad+bc)/_L\psi(bd)$, be definition of ψ on F,
$= (ad+bc)/_L(bd)$, since ψ is the identity on D,
$= a/_L b + c/_L d = \psi(a/_F b) + \psi(c/_F d) = \psi(x) + \psi(y)$.

Finally, to show ψ is one-to-one, suppose $\psi(a/_F b) = \psi(c/_F d)$. Then $\psi(a)/_L\psi(b) = \psi(c)/_L\psi(d)$ and $\psi(a)\psi(d) = \psi(b)\psi(c)$ or (since ψ is the identity on D) $ad = bc$. Therefore, $a/_F b = c/_F d$ and so ψ is one-to-one. Therefore ψ is an isomorphism, as desired. $\square$

Theorem 21.6 (continued 2).

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Corollary 21.8.

Corollary 21.8. Every field L containing an integral domain D contains a field of quotients of D.

Proof. From the proof of Theorem 21.6, we know that F is a field and ψ is a ring (and field) isomorphism, so $\psi[F]$ is a field. As seen in the proof, $\varphi[F]$ is a field of quotients of elements of D subfield of D. $\square$

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Corollary 21.9.

Corollary 21.9. Any two fields of quotients of an integral domain are isomorphic.

Proof. Suppose L is a field of quotients of D. Then every element $x \in L$ is of the form $x = a/_L b$ for some $a, b \in D$. So $L \subseteq \psi[F]$ using the notation of Theorem 21.6, and similarly $\psi[F] \subseteq L$. So $\psi[F] = L$ and the two fields of quotients F and L are isomorphic. $\square$

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