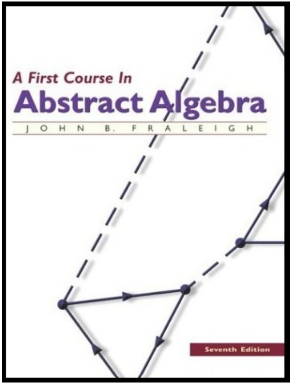
Introduction to Modern Algebra

Part VI. Extension Fields

VI.32. Geometric Constructions



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Corollary 32.5

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Proof. By Theorem 32.1, the constructible numbers C satisfy (1) $0 \in C$ (a point is a line segment of length 0), (2) $\alpha - \beta \in C$ for all $\alpha, \beta \in C$, and (3) $\alpha\beta \in C$ for all $\alpha, \beta \in C$. So, by Exercise 18.48, C is a subring of $\mathbb{R}$. Commutativity of multiplication in C is inherited from $\mathbb{R}$. Since $\alpha = 1 \in C$ by definition, then Theorem 32.1 implies $\alpha/\beta = 1/\beta \in C$ for all $\beta \neq 0$, so C is a division ring. That is, C is a field. □

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Theorem 32.9. Doubling the Cube is Impossible

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Proof. We only need a counterexample to the doubling the cube problem. Suppose a cube has a side of the length of the given unit 1. Then the volume of the cube is 1. The desired cube then has volume 2 and sides of length $\sqrt[3]{2}$. But $\sqrt[3]{2}$ is a zero of $x^3 - 2$ (irreducible in $\mathbb{Q}$) and so $\deg\left(\sqrt[3]{2}, \mathbb{Q}\right) = 3$ and as in if $\gamma = \sqrt[3]{2}$ is constructible then.

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Theorem 32.10. Squaring the Circle is Impossible

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Proof. Consider a circle of radius the given unit 1. The are of this circle is π. So the desired square would have a side of length $\sqrt{\pi}$. But π is transcendental over $\mathbb{Q}$ (as shown by Ferdinan Lindemann in 1882; see page 298) and so $\sqrt{\pi}$ is transcendental over $\mathbb{Q}$. Hence $\sqrt{\pi}$ is not algebraic and not constructible. □

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Theorem 32.11. Trisecting the Angle is Impossible

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Proof. In the supplement, we show that angle θ is constructible if and only if length $|\cos\theta|$ is constructible (see also Figure 32.12). Now 60° is constructible since an equilateral triangle is constructible (Euclid's Elements of Geometry, Book I, Proposition 1). We now use a trigonometric identity to show that a 60° angle cannot be trisected. Recall from the summation formula for $\cos\theta$ that $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$. Let $\theta = 20^\circ$ and then $\cos(3\theta) = \cos(60^\circ) = \frac{1}{2}$. Let $\alpha = \cos(20^\circ)$. Then $\frac{1}{2} = 4\alpha^3 - 3\alpha$ or $8\alpha^3 - 6\alpha - 1 = 0$. So α is a zero of $p(x) = 8x^3 - 6x - 1$.

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Theorem 32.11. Trisecting the Angle is Impossible (Continued).

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Proof (Continued). However if α is construcible then by Corollary 32.8 we need $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2^r$ for some integer $r \geq 0$. So $\alpha = \cos(20^\circ)$ is not constructible and hence $20^\circ = \theta/3$ is not constructible. □