Groups: A Geometric Introduction

A Presentation Given to the ETSU Abstract Algebra Club

SLIDE. Groups: A Geometric Introduction. By Robert "Dr. Bob" Gardner.

SLIDE. A core course in any undergraduate mathematics education is an introduction to modern algebra. At ETSU, this is the cross listed undergraduate/graduate class MATH 4127/5127. The book we currently use (currently being the academic year 2014-2015) is John B. Fraleigh's *A First Course in Abstract Algebra*, seventh edition, published by Addison-Wesley in 2002. In fact, *I* used the third edition of this book in *my* undergraduate algebra classes. In about the third week of a modern algebra class, you are introduced to the central concept of a "group."

SLIDE. GROUPS.

SLIDE. Fraleigh defines a group in Section I.4. Formally, a group is a set G of elements, along with a binary operation which we now denote with a star. A binary operation is simply a way of taking two elements of G and producing an element of G associated with the two (ordered) elements. An example is the set of integers along with the binary operation of addition. The integer 7 is associated with the two elements 3 and 4 since $3+4=7$. We require that the group G be *closed* under binary operation star—that is, we require that the element produced by the binary operation is itself in the group. This is why we cannot use the set of integers along with the binary operation of division as a group since, for example, 3 *divided* by 4 is not an integer. In addition, we require three properties (or "axioms," if you like). $\mathcal{G}_1$: We require the binary operation star to be associative. That is, for all $a, b, c \in G$ we have $(a*b)*c = a*(b*c)$. $\mathcal{G}_2$: We require the existence of an *identity* element of a group—that is, an element e in G such that for all x in G we have $e*x = x*e = x$. The identity element under addition is 0 and the identity element under multiplication is 1. $\mathcal{G}_3$: We require that each a in G has an *inverse* in G, which we denote here as a'—that is, we require $a * a' = a' * a = e$. Under addition, the inverse of a number is the negative of that number. Under multiplication, the inverse of a number is the reciprocal of that number.

SLIDE. You are already familiar with several groups. For example, the integers under addition, the rational numbers under addition, the real numbers under addition, the complex numbers under addition, and the slightly more exotic example of the integers modulo n. **(Enter)** Associativity is clear in each of these examples. Since the groups are additive, 0 is the identity, and the inverse of any x is $-x$. Of course modulo n, the inverse of x is $n - x$.

SLIDE. Some other familiar groups include the nonzero rational numbers under multiplication, the nonzero real numbers under multiplication, and the nonzero complex numbers under multiplication. **(Enter)** Again, associativity is clear. Since these are multiplicative groups, 1 is the identity and the inverse of any x is the reciprocal of x.

SLIDE. In the definition of a group, we did not require commutativity of the binary operation. However, each of the above examples are based on binary operations which *are* commutative. Such groups are called *abelian groups*. **(Enter)** In your sophomore year, you encountered a noncommutative binary operation when you dealt with the multiplication of square matrices. So an example of a non-abelian group is the set of all invertible square matrices under matrix multiplication.

SLIDE. Associativity is "clear" in each of the previous examples. In your undergraduate career, you are unlikely to encounter many non-associative binary operations. However, the cross product of vectors in $\mathbb{R}^3$ which you see in Linear Algebra and Calculus 3 is not associative. **(Enter)** That is, we do not in general have that $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \times \vec{w}$. We can show this by considering a specific example. We have from the standard unit vectors $\hat{\imath}$ and $\hat{\jmath}$ in $\mathbb{R}^3$ that $\hat{\imath} \times (\hat{\imath} \times \hat{\jmath}) = -\hat{\jmath}$ but $(\hat{\imath} \times \hat{\imath}) \times \hat{\jmath} = \vec{0}$. **(Enter)** So we cannot form a group using the cross product as the binary operation since this operation is not associative!

SLIDE. SYMMETRY GROUPS

SLIDE. Each of the above examples of groups are "algebraic." We want a more geometric—a more *visual*—way to illustrate groups. **(Enter)** We do so by defining an *isometry*. An isometry of n-dimensional space $\mathbb{R}^n$ is a function from $\mathbb{R}^n$ onto $\mathbb{R}^n$ that preserves distance. **(Enter)** More precisely, the mapping π is an isometry from $\mathbb{R}^n$ to $\mathbb{R}^n$ if for all $x, y \in \mathbb{R}^n$ we have the distance from x to y is the same as the distance from $\pi(x)$ to $\pi(y)$. We see this expressed here using a *metric* d on $\mathbb{R}^n$.

SLIDE. Now for the definition of a *symmetry group*. Let F be a set of points in $\mathbb{R}^n$. The symmetry group of F in $\mathbb{R}^n$ is the set of all isometries of $\mathbb{R}^n$ that carry F onto itself. The group operation is function composition. For our examples, n will usually be 1, 2, or 3 so that we can visualize the action of the symmetry group on object F.

SLIDE. Consider the line segment I in $\mathbb{R}^1$ from $a = 0$ to $b = 1$. We'll see that the symmetry group of I is affected by the dimension of the space in which we consider it to be. There are two isometries of $\mathbb{R}^1$ which map I onto itself: $f_0(x) = x$ and $f_1(x) = 1-x$. **(Enter)** f_0 is the identity isometry and f_1 "flips" the real line about the point $x = 1/2$. Latter, we will call this type of flip a *reflection*. **(Enter)** The "multiplication table" (sometimes called a "Cayley table") for the symmetry group of interval I is as given here. Since this group only has two elements, it is said to be of *order* 2.

SLIDE. We now consider the same line segment I, but as a subset of $\mathbb{R}^2$ instead of $\mathbb{R}^1$. This changes the symmetry group of I. It now has four elements. They are the functions (1) $f_0(x, y) = (x, y)$ which is the identity function, (2) $f_1(x, y) = (1-x, y)$ which is reflection about the line $x = 1/2$, (3) $f_2(x, y) = (x, -y)$ which is reflection about the line $y = 0$, and (4) $f_3(x, y) = (1 - x, -y)$ which is a combination of f_1 and f_2.

SLIDE. Let's illustrate each of the isometries. Here we see the x, y-plane with the quadrants color coded and segment I represented with a red line segment. **(Enter)** f_0 is the identity and leaves everything unchanged. **(Enter)** f_1 is a reflection about the line $x = 1/2$ and so it flips interval I over and swaps its endpoints. This moves the second quadrant (in orange) and the third quadrant (in green) to the far right, while moving the first quadrant (in yellow) and the fourth quadrant (in blue) to the left. **(Enter)** f_2 is a reflection about the x-axis and so simply interchanges the upper half-plane and the lower half-plane. **(Enter)** f_3 is a combination of two reflections, one about the line $x = 1/2$ and the other about the x-axis. This moves the first quadrant (in yellow) to the lower left, the second quadrant (in orange) to the lower right, the third quadrant (in green) to the upper right, and the fourth quadrant (in blue) to the upper left. **(Enter)** This symmetry group is denoted D_2 and has this multiplication table. Notice that, for example, f_2 composed with itself (that is, $f_2 \circ f_2$) is the identity. This makes geometric sense, since by reflecting the plane about the x-axis twice puts it back into its original position. In fact, each of the elements of this group is its own inverse.

SLIDE. Now for something more interesting. We consider the symmetries of an equilateral triangle. To keep track of the orientation of the triangle, we label the vertices of the triangle with black numbers 1, 2, and 3. We label the original location of the vertices with red numbers 1, 2, and 3. **(Enter)** We introduce a permutation notation that tells us how the black numbers correspond to the red numbers. So for the identity permutation, which we denote as ρ_0, we have the black 1 corresponding to the red 1, the black 2 corresponding to the red 2, and the black 3 corresponding to the red 3. **(Enter)** Now we rotate the triangle 120° clockwise and now we have the black 1 corresponding to the red 2, and so forth as given here. We denote this correspondence as ρ_1. **(Enter)** If we rotate the triangle another 120° (and so we have rotated it 240° from its original orientation) then we have the correspondence as given here which we denote as ρ_2. **(Enter)** If we rotate the triangle through 120° again, then it returns to its original orientation and we are back to the identity ρ_0.

SLIDE. We can also reflect the triangle about an axis. **(Enter)** If we reflect it about a vertical axis then we have the black and red 1's corresponding as they did initially, and the black 2 and black 3 interchanged. We denote this permutation μ_1. We can similarly fix one of the other vertices and interchange the remaining two. **(Enter)** We can fix vertex 2 and interchange vertices 1 and 3, denoted μ_2. **(Enter)** We can fix vertex 3 and interchange vertices 1 and 2, denoted μ_3.

SLIDE. Now we compose two of these permutations by performing one and then the next. **(Enter)** Suppose we first apply permutation μ_1. **(Enter)** Now we follow that with permutation ρ_1. This produces the product $\rho_1 * \mu_1$ and we see that it gives μ_3. Here's how we read the product of two permutations. We read from right to left. So to find where the black 1 is mapped, notice that it first is mapped to 1 by μ_1 and then 1 is mapped to the red 2 by ρ_1. So in the product, the black 1 is mapped to the red 2. Similarly, the black 2 is mapped to 3 and then to the red 1; the black 3 is mapped to 2 and then to the red 3. So the product $\rho_1 * \mu_1$ is the permutation μ_3.

SLIDE. Let's compute a product again using ρ_1 and μ_1, but in the opposite order. **(Enter)** First apply ρ_1. **(Enter)** Now apply μ_1. This gives us that the product $\mu_1 * \rho_1$ is μ_2. Therefore we see that $\mu_1 * \rho_1 \neq \rho_1 * \mu_1$ and so the symmetry group of the triangle is not an abelian group; that is, it is not a commutative group.

SLIDE. This group of symmetries is called the dihedral group D_3. Here is the multiplication table for D_3. Since the entries in this table are not symmetric with respect to the main diagonal, we again see that the group is not abelian. **(Enter)** Notice that the rotations alone (that is, the ρ's) form a *subgroup* of order 3.

SLIDE. In fact, instead of using an equilateral triangle, we could use any regular polygon in the discussion of symmetry groups. The symmetry group of the regular n-gon is the dihedral group D_n which is of order $2n$. **(Enter)** These groups are *generated* by two fundamental permutations: rotations and reflections.

SLIDE. If we consider the symmetries of a regular n-gon which only consists of the rotations (and not the reflections) then we get a subgroup of the dihedral group D_n which consists of the n rotational permutations. **(Enter)** This group of n rotational permutations forms the *cyclic* group of order n. The cyclic group of order n is the same (that is, it is "isomorphic to") the integers modulo n. **(Enter)** Since, for each n, the cyclic group of order n is a subgroup of the dihedral group of order $2n$, we often drop the binary operation and write this as $\mathbb{Z}_n < D_n$.

SLIDE. As an example, consider a regular hexagon. **(Enter)** There are six rotational permutations. Notice that these permutations map 1 to: 1, then 2, then 3, 4, 5, 6, and back to 1. So in this cyclic group, 1 cycles around 6 times back to itself (as do all the other numbers).

SLIDE. In the cyclic group $\mathbb{Z}_n$, every element can be "generated" by the first rotation ρ_1. That is, each element of $\mathbb{Z}_n$ is a power of ρ_1: $\rho_2 = \rho_1 * \rho_1$, $\rho_3 = \rho_1 * \rho_1 * \rho_1$, and so forth. In fact, the formal definition of a *cyclic* group is a group which is generated by a single element. **(Enter)** The dihedral group D_n can be generated by two symmetries: a rotation and a reflection. *This* is the reason these groups are called *di*hedral groups. They are generated by two ("di" for two) elementary permutations.

SLIDE. It should be no surprise that we want to distinguish between finite and infinite groups. A group G with binary operation $*$ is *finite* if the set G has a finite number of elements. **(Enter)** We can easily classify the finite symmetry groups of $\mathbb{R}^2$. By a theorem on page 115 of Fraleigh, we have: The only finite symmetry groups of a set of points in $\mathbb{R}^2$ (that is, the only "plane symmetry groups" or "groups of isometries of the plane") are the groups $\mathbb{Z}_n$ and D_n for some n. These groups are sometimes called *rosette groups*. **(Enter)** Since we have classified the finite plane symmetry groups, we now turn our attention to the infinite plane symmetry groups. This is covered briefly in Fraleigh's Section II.12, but is covered in some detail in Chapter 28 of Joseph Gallian's *Contemporary Abstract Algebra*, 8th edition.

SLIDE. PLANE ISOMETRIES

SLIDE. To explore the infinite plane symmetry groups, we only need to look at four types of isometries. Let F be a set of points in $\mathbb{R}^2$. A translation τ of F is a rigid movement of F in some direction, say the direction given by vector $\vec{v} = [v_1, v_2]$. The image of F is then $\tau(F)$, the set of all points $(x, y) \in F$ which have first coordinate $x + v_1$ and second coordinate $y + v_2$. **(Enter)** We represent set F as a square. **(Enter)** Here is vector $\vec{v}$. **(Enter)** So $\tau(F)$ is just a rigid translation of F by vector $\vec{v}$.

SLIDE. A rotation ρ of F about a point P through an angle θ is simply a rotation of each point of F through an angle θ about point P. **(Enter)** Here is angle θ. **(Enter)** And here is the rotation $\rho(F)$.

SLIDE. A reflection μ of F across a line ℓ is a mapping of each point of F to its mirror image with respect to line ℓ. This time we represent set F as a winking smiley face. We use a winking smiley face instead of a square because the winking smiley face, unlike the square, has no internal symmetries. **(Enter)** $\mu(F)$ is then given as shown here. Notice the mirror image effect. The winking eye was originally on the right in F, but it is on the left in $\mu(F)$.

SLIDE. A glide reflection γ of F is a combination of a translation and a reflection across a line ℓ. This produces a translated mirror image. **(Enter)** So first F is translated by vector $\vec{v}$ and then reflected about line ℓ. Notice that we could have done the reflection first and the translation second and we would have ended up with the same image $\gamma(F)$.

SLIDE. Translations and rotations are examples of *orientation-preserving* isometries (which we illustrated with squares), whereas reflections and glide reflections are examples of *orientation-reversing* motions (which we illustrated with winking smiley faces). **(Enter)** In fact, the four isometries listed above are the only isometries of the plane. As a theorem, this result is: Any isometry of the plane, $\mathbb{R}^2$, is either a translation, rotation, reflection, or glide reflection.

SLIDE. Notice that the group $\mathbb{Z}_n$ consists only of rotations. The group D_n only consists of rotations and reflections. That is, the finite groups of symmetries of $\mathbb{R}^2$ do not include any translations nor any glide reflections.

SLIDE. THE FRIEZE GROUPS

SLIDE. We now consider the "discrete frieze groups." A discrete frieze group consists of a pattern of finite width and height that is repeated endlessly in both directions along its baseline to form a strip of infinite length by finite height. **(Enter)** For example, we might have the finite pattern and baseline given here. The baseline is not actually part of the pattern, but is just used to illustrate how the finite pattern is repeated. **(Enter)** This produces the infinite pattern given here.

SLIDE. This pattern of F's clearly has a symmetry which admits translations only. **(Enter)** We can translate to the right one step—we associate this with integer 1. **(Enter)** We can translate to the right two steps—we associate this with integer 2. **(Enter)** We can translate to the right three steps—we associate this with integer 3. **(Enter)** We can translate to the right four steps—we associate this with integer 4. **(Enter)** Starting over, we can translate to the left one step—we associate this with integer -1. **(Enter)** We can translate to the left two steps—we associate this with integer -2. **(Enter)** So for each integer there is a corresponding translation (and conversely; the identity translation corresponds to 0). So the frieze group associated with this pattern is the integers (under addition).

SLIDE. This pattern of D's has the translational symmetry which the pattern of F's had. But it also admits a rotational symmetry about a horizontal line through the center of the pattern. **(Enter)** We can translate to the right by 1 with no rotation—we associate this with the pair $(1,0)$. **(Enter)** We can translate to the left by 1 with no rotation—we associate this with the pair $(-1,0)$. **(Enter)** We can have no translation and a rotation—we associate this with the pair $(0,1)$.

SLIDE. We can translate to the right by 1 with 1 rotation—we associate this with the pair $(1, 1)$. **(Enter)** We can translate to the left by 1 with 1 rotation—we associate this with the pair $(-1, 1)$. **(Enter)** So for each pair (x, y) where x is an integer and y is either 0 or 1, there is a corresponding translation and rotation combination which is a symmetry of the pattern. The symmetry group of this pattern is $\mathbb{Z} \times \mathbb{Z}_2$.

SLIDE. This pattern of T's has the translational symmetry which the pattern of F's had. But it does not admit the rotational symmetry which the pattern of D's had. However, it does admit rotational symmetries about a vertical axis. **(Enter)** We can translate to the right by 1. **(Enter)** We can rotate about a vertical axis half way between two of the T's. **(Enter)** We can rotate about a vertical axis through the middle of one of the T's. **(Enter)** The symmetry group consists of combinations of these symmetries. The symmetry group is denoted D_∞.

SLIDE. This pattern of T's admits all of the symmetries of the previous pattern of T's. **(Enter)** We can translate to the right by 1. **(Enter)** We can rotate about a vertical axis half way between two of the T's. **(Enter)** We can rotate about a vertical axis through the middle of one of the T's. **(Enter)** So the symmetry group includes all of the symmetries in D_∞.

SLIDE. But this pattern of T's also admits a rotation about a **(Enter)** horizontal axis. The symmetry group is $D_\infty \times \mathbb{Z}_2$.

SLIDE. We have now seen an example of each of the frieze groups. As a theorem we have: The collection of discrete frieze groups consist precisely of the following four groups: (1) the integers, (2) the integers times $\mathbb{Z}_2$, (3) the infinite dihedral group D_∞, and (4) the infinite dihedral group times $\mathbb{Z}_2$. We observe that since the integers are a subgroup of D_∞, then each frieze group contains the integers as a subgroup.

SLIDE. Even though there are only four discrete frieze groups, there are seven types of "frieze patterns." Gallian describes these seven patterns schematically as follows. **(Enter)** Pattern 1. **(Enter)** The frieze group is the integers and is generated by a translation.

SLIDE. (Enter) Pattern 2. **(Enter)** The frieze group is the integers. It is generated by a glide reflection. **(Enter)** Pattern 3. **(Enter)** The frieze group is D_∞. It is generated by a translation and a reflection about a vertical line.

SLIDE. (Enter) Pattern 4. **(Enter)** The frieze group is D_∞. It is generated by a translation and a rotation about a point. **(Enter)** Pattern 5. **(Enter)** The frieze group is D_∞. It is generated by a glide reflection and a reflection about a vertical line.

SLIDE. (Enter) Pattern 6. **(Enter)** The frieze group is $\mathbb{Z}\times\mathbb{Z}_2$. It is generated by a translation and a reflection about a horizontal line. **(Enter)** Pattern 7. **(Enter)** The frieze group is $D_\infty \times \mathbb{Z}_2$. It is generated by a translation, a reflection about a vertical line, and a reflection about a horizontal line.

SLIDE. There are some more interesting or artistic illustrations of the frieze patterns then a pattern of R's. Here are John Conway's Frieze Pattern Dance Steps.

SLIDE. There is a flowchart to determine which of the seven frieze patterns applies based on which symmetries the pattern admits.

SLIDE. Here is the pattern of the trim on my wife's china cabinet. **(Enter)** Here is the idealized version of the pattern. **(Enter)** It admits a translation. **(Enter)** It admits a rotation about a horizontal axis. **(Enter)** It admits a rotation about a vertical axis. **(Enter)** Therefore the frieze group is $D_\infty \times \mathbb{Z}_2$.

SLIDE. THE WALLPAPER GROUPS

SLIDE. The frieze patterns include regular repetitions in one direction (or "dimension") of a fundamental pattern. So the frieze groups include all powers of just a translation—this is why $\mathbb{Z}$ is a subgroup of each frieze group. **(Enter)** If we consider regular repetitions in two directions (or "dimensions") of a fundamental pattern, then we will get a symmetry group which includes all powers of two (independent) translations. The symmetry group will then include $\mathbb{Z} \times \mathbb{Z}$ as a subgroup. Such a symmetry group is called a *crystallographic group* or a *wallpaper group.*

SLIDE. Consider the grid of R's. We can translate the pattern left/right, up/down, or a combination of these. **(Enter)** We can translate to the right by 1—we associate this with the pair $(1, 0)$. **(Enter)** We can translate to the left by 1—we associate this with the pair $(-1, 0)$. **(Enter)** We can translate up by 1—we associate this with the pair $(0, 1)$. **(Enter)** We can translate down by 1—we associate this with the pair $(0, -1)$.

SLIDE. (Enter) We can translate to the right by 2 and up by 1—we associate this with the pair $(2, 1)$.

SLIDE. We can translate to the left by 2 and down by 1—we associate this with the pair $(-2, -1)$. **(Enter)** Since this pattern admits no rotations nor reflections, then the symmetry group is $\mathbb{Z} \times \mathbb{Z}$ (this wallpaper group is denoted $p1$).

SLIDE. The symmetry group can be $\mathbb{Z} \times \mathbb{Z}$, yet the grid of images not arranged in rows and columns at right angles. **(Enter)** Here in Makato Nakamura's Fish3 tessellation of the plane, **(Enter)** the grid of images is generated by translations in these directions. **(Enter)** These directions determine a *unit* or *cell*. **(Enter)** We view the plane as tiled by the unit. The coloration here should be ignored; it is just for an artistic effect.

SLIDE. If we pick a point in the plane and then follow that point as it is translated by an integer amount in the two principle directions, **(Enter)** then we get a lattice of points.

SLIDE. One can show that there are only five types of lattices that can occur in a wallpaper group.

SLIDE. Based on these lattices, it can be shown that there are 17 wallpaper groups. Here are the 17 lattice units with the reflections and rotations represented symbolically. In the first lattice, there are no rotations or reflections so the associated wallpaper group, denoted $p1$, must in fact just be the group of translations. But the group of translations is isomorphic to $\mathbb{Z} \times \mathbb{Z}$, so we have that $p1$ and $\mathbb{Z} \times \mathbb{Z}$

are isomorphic. We now discuss one of the examples in more detail and illustrate how to generate a wallpaper pattern.

SLIDE. The previous two slides contained images from Doris Schattischneider's paper "The Plane Symmetry Groups: Their Recognition and Notation" which appeared in the *American Mathematical Monthly* in 1978. Much of this presentation and much of Gallian's chapter on wallpaper groups is based on this paper. Here is a part of Chart 5 from Schattischneider's paper. It gives the lattice unit and symmetries for the group $p31m$. The shaded region is the generating region. By filling it with an image, we will determine the rest of the wallpaper pattern by using the symmetries. The small equilateral triangle represents an order 3 rotation. That is, we will rotate the generating region 120° about the triangle and 240° about the triangle. The double lines represent an axis of reflection so we will reflect about this axis. Then we will apply the translations, which are given by the arrows.

SLIDE. We start with the generating region and the symmetries. **(Enter)** First we introduce a pattern with no symmetries; we use our old friend the winking smiley face. **(Enter)** Rotate the region through 120°. **(Enter)** Rotate the region through another 120°. **(Enter)** Rotating through a third 120°, of course, returns the winking smiley face to its original position. This is the reason the rotational symmetry is said to be of order 3. **(Enter)** Reflect about the axis of reflection. This completes the lattice unit.

SLIDE. Let's clean up the lattice unit. Now for the translations. **(Enter)** We translate the unit to the right. **(Enter)** ... and to the left. **(Enter)** We translate the unit negative one times "upward" (as we will call the non-horizontal direction).

(Enter) . . . negative one time upward and to the left. **(Enter)** . . . negative one time upward and to the right. **(Enter)** Continuing, we tile the plan with the lattice unit.

SLIDE. We now drop the outline of the lattice units. This gives us our wallpaper pattern. Can you see the symmetries? **(Enter)** To see the order 3 rotation, we need to consider a collection of four of the smiley faces. **(Enter)** These four smiley faces can be rotated to reveal the rotational symmetry. **(Enter)** The axes of reflectional symmetries are given here. **(Enter)** These circles of smiley faces are suggestive of the translational symmetry. **(Enter)** By introducing a lattice of circles, we can see how the pattern can be translated left and right and upward and downward.

SLIDE. Now we start with a pattern and try to deduce its symmetry group. Here is the brick work of my fireplace. **(Enter)** Here is the pattern idealized.

SLIDE. There is a horizontal axis of reflection. **(Enter)** There is a vertical axis of reflection. **(Enter)** There is a horizontal axis of glide reflection. **(Enter)** There is a vertical axis of glide reflection. **(Enter)** There are several order 2 rotations. **(Enter)** Here is a lattice unit. **(Enter)** This is the same as the lattice unit with symmetries for group *cmm*. So the group of symmetries of this brick pattern is group *cmm*.

SLIDE. There is also a flowchart for determining a symmetry group for a wallpaper pattern. Let's apply this to the brick pattern. **(Enter)** The largest order of rotation? **(Enter)** We know that the brick pattern has only order 2 rotations, which are indicated by the red diamonds. **(Enter)** Is there a reflection? **(Enter)** Yes, there is, for example, a horizontal axis of reflection. **(Enter)** Are there reflections in 2 directions? **(Enter)** Yes, there are both horizontal and vertical axes of reflection. **(Enter)** Are all rotation centers on mirror lines (that is, on axes of reflection)? **(Enter)** No, these four centers of rotation are not on axes of reflection. **(Enter)** So, again, the symmetry group is *cmm*.

SLIDE. THE CRYSTALLOGRAPHIC GROUPS IN HIGHER DIMENSIONS

SLIDE. We can consider symmetry groups in higher dimensional spaces as well. We can classify the symmetry groups in three dimensions as follows. **(Enter)** The only finite symmetry groups of a set of points in $\mathbb{R}^3$ (that is, the only "finite groups of isometries of 3-space") are the groups $\mathbb{Z}_n$ (for some n), D_n (for some n), S_4, A_4, and A_5. You will encounter the symmetry groups S_n and their subgroups, the alternating groups A_n, in Introduction to Modern Algebra. **(Enter)** There are a total of 230 crystallographic groups in 3-dimensions. In 4-dimensions, there are 4,783 crystallographic groups.

SLIDE. Now we consider the platonic solid with four sides called a tetrahedron. To do so, we take the perspective of looking straight down on the top vertex. **(Enter)** As in two dimensions we label the vertices on the tetrahedron in black and introduce red numbers outside of the tetrahedron to track the symmetries. First, we start with the identity permutation iota. **(Enter)** Now with vertex 1 fixed, we rotate the tetrahedron 120° clockwise thus mapping the black 2 to the red 3, the black 3 to the red 4, and the black 4 to the red 2. We call this permutation τ_1. **(Enter)** If we rotate through another 120° then we get this permutation which we call τ_2. **(Enter)** Of course another 120° takes us back to the identity iota.

SLIDE. We now take a perspective with vertex 2 at the top. **(Enter)** We then get the identity and two new permutations.

SLIDE. Taking the perspective with vertex 3 at the top, **(Enter)** we get the identity and two more permutations.

SLIDE. With vertex 4 at the top, **(Enter)** we again get the identity and two more permutations.

SLIDE. There is another type of symmetry of the tetrahedron. **(Enter)** With the appropriate axis, we can rotate the tetrahedron around in such a way as to interchange vertices 1 and 2 and interchange vertices 3 and 4.

SLIDE. (Enter) Similarly, we can interchange vertices 1 and 4 and interchange vertices 2 and 3.

SLIDE. For the final symmetry, we view the tetrahedron from a slightly different perspective. **(Enter)** We can interchange vertices 1 and 3 and interchange vertices 2 and 4.

SLIDE. With the notation we have introduced (which comes from the *Schaum's Outline of Theory and Problems of Group Theory*) we have the following multiplication table for the group of symmetries of the tetrahedron. This group is denoted A_4. **(Enter)** Notice that this group has a subgroup of order 4. It is the subgroup which consists of the identity and all the sigma permutations (these are the ones which interchanged pairs of vertices). In fact, there is an interesting behavior to the distribution of the permutations in this table. **(Enter)** This array of colors has the condition that all of the sigmas are in a yellow square, the elements τ_1, τ_4, τ_5, and τ_8 are in green squares, and τ_2, τ_3, τ_6, and τ_7 are in orange squares. These large 4 by 4 colored squares are *cosets* of the subgroup of sigmas. In fact, these cosets themselves form a group of order 3. This group of cosets is isomorphic to the integers modulo 3, $\mathbb{Z}_3$.

SLIDE. The Platonic solids are regular polyhedra with each face a regular polygon. The five Platonic solids are the tetrahedron, cube, octahedron, dodecahedron, and icosahedron. **(Enter)** The symmetry groups for the Platonic solids are: the tetrahedron has symmetry group A_4, the cube has symmetry group S_5, the octahedron also has symmetry group S_5, the dodecahedron has symmetry group A_5, and the icosahedron also has symmetry group A_5. **(Enter)** The cube and octahedron are *duals*; the dodecahedron and icosahedron are duals. The tetrahedron is self dual. This duality relationship is the reason the symmetry groups pair up as they do.

SLIDE. A topic covered in Introduction to Modern Algebra is the topic of simple groups. The mathematical community spent over 30 years trying to classify finite simple groups. In this exploration, several sporadic groups of very large order were discovered. The largest one is called the monster. It is roughly of order 8.1×10^{53}. **(Enter)** This group was predicted to exist in 1973 by Berndt Fischer and Robert Griess and constructed by Griess in 1982. **(Enter)** Wikipedia describes the monster as the automorphism group of the Griess algebra, which is a 196,884-dimensional commutative nonassociative algebra. The important thing to take from this is that the monster is a symmetry group of an object in 196,884-dimensional space. So these symmetry groups are prolific!

SLIDE. ALGEBRA?

SLIDE. A very reasonable question is: "What the hell does all this stuff have to do with the quadratic equation and polynomials? You know, classical algebra!" **(Enter)** Groups (and rings and fields) became part of algebra in the 19th century. While looking for an algebraic formula for the zeros of an nth degree polynomial (like a quadratic equation for an nth degree polynomial), Abel showed that there is not (in general) an algebraic solution to a 5th degree polynomial equation. Galois gave conditions for the existence of an algebraic solution of a general nth degree polynomial equation. These conditions involved *permutations* of the zeros of the polynomial. **(Enter)** These permutations of the zeros of a polynomial in the study of algebraic equations led in the 19th century to the more abstract study of groups in general and permutation groups in particular. By the middle of the 20th century, group theory and "abstract algebra" had become a major area of mathematics. **That's** why you take modern algebra as an undergraduate!

SLIDE. REFERENCES

SLIDE. WEBSITES