Introduction to Algebra, MATH 5127

Homework 10, Sections IV.18 and IV.19, Solutions

Due Friday April 19, 2013 at 2:30

18.20. Consider the ring $M_2(\mathbb{Z}_2)$. **(a)** Find the order of the ring (that is, the number of elements in it). **(b)** Find all units in $M_2(\mathbb{Z}_2)$. *Explain* your answer by checking each matrix; do not simply list the units.

18.37. Let U be the set of all units in ring $\langle R, +, \cdot \rangle$ with unity. Prove $\langle U, \cdot \rangle$ is a group. HINT: Show closure of U under $\cdot$, show $\langle U, \cdot \rangle$ satisfies $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, and mention when you use $\mathcal{R}_1$, $\mathcal{R}_2$, or $\mathcal{R}_3$.

18.A. Let $\langle R, +, \cdot \rangle$ be a ring. The *center* of R is the set $S = \{a \in R \mid ar = ra \text{ for all } r \in R\}$. Prove the center of a ring is a subring. HINT: You may use Exercise 18.48 concerning subrings, but justify computations by quoting the use of $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$, or any theorems.

18.B. Let $\langle R, +, \cdot \rangle$ be a commutative ring with unity. Suppose a is a unit in R and $b^2 = 0$. Prove that $a + b$ is a unit in R. HINT: Use long division (in your scratch work) to find "$1/(a + b)$" and then show that $a + b$ is a unit with a computation. JUSTIFY ALL STEPS IN YOUR COMPUTATIONS!

19.12 Let $\langle R, +, \cdot \rangle$ be a commutative ring with unity and characteristic 3. Since R is commutative, the Binomial Theorem holds. Use it to compute and simplify $(a + b)^9$ for $a, b \in R$.

19.A. Let $\langle R, +, \cdot \rangle$ be a commutative ring with nonzero $a, b \in R$ and let ab be a zero divisor. Prove that either a or b is a zero divisor.

19.B. Let $\langle R, +, \cdot \rangle$ be a finite commutative ring with no zero divisors and with at least two elements. Prove that R has unity. HINT: Let $R = \{0, r_1, r_2, \ldots, r_n\}$ where $r_i \neq 0$. Show $r_1 R = R$. Conclude that some r_j is unity.