Introduction to Algebra, MATH 5127

Homework 12, Sections IV.22 and IV.23

Due Friday May 3, 2013 at 2:30

22.15. Find all zeros of $f(x)g(x)$ in $\mathbb{Z}_7$ where $f(x) = x^3 + 2x^2 + 5$ and $g(x) = 3x^2 + 2x$. Factor polynomial $f(x)g(x)$ as much as possible using the Factor Theorem (Corollary 23.3). Remember that all arithmetic is done in $\mathbb{Z}_7$ and so all coefficients you compute must be in $\{0, 1, 2, 3, 4, 5, 6\}$.

22.24. Prove that if D is an integral domain, then $D[x]$ is an integral domain. HINT: Show that $D[x]$ is commutative, has unity, and contains no zero divisors.

22.27a. Let F be a field of characteristic zero and let D be the formal polynomial differentiation map

$$D(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) = a_1 + 2a_2 + \cdots + na_nx^{n-1}.$$

(1) Show that $D : F[x] \to F[x]$ is a group homomorphism of $\langle F[x], +\rangle$ into itself. (2) Is D a group automorphism of $\langle F[x], +\rangle$ with itself? (3) Is D a ring homomorphism of $\langle F[x], +, \cdot\rangle$ with itself?

23.2. For $f(x) = x^6 + 3x^5 + 4x^2 + 4x + 2$ and $g(x) = 3x^2 + 2x + 4$ in $\mathbb{Z}_7[x]$, find $q(x)$ and $r(x)$ guaranteed by the Division Algorithm (Theorem 23.1). HINT: In $\mathbb{Z}_7[x]$, all coefficients are from $\{0, 1, 2, 3, 4, 5, 6\}$.

23.9. The polynomial $f(x) = x^4 + 4$ can be factored into linear factors in $\mathbb{Z}_5[x]$. Find the factorization by finding the zeros of $f(x)$.

23.A. Let $f(x) \in \mathbb{Z}_p[x]$ for prime p. Prove that if $f(b) = 0$ for $b \in \mathbb{Z}_p$ and $b \neq 0$. Then $f(b^p) = 0$.