Introduction to Algebra, MATH 5127

Homework 3, Sections I.2 and I.3

Due Friday February 8, 2013 at 2:30

I.2.11. Determine whether binary operation $*$ defined on $\mathbb{Z}^+ = \mathbb{N}$ as $a * b = a^b$ is commutative and whether it is associative. If $*$ does satisfy one of these properties, then you need to give a (general) proof. If it fails one of these properties, then a specific example which illustrates this failure is sufficient.

I.2.23b. Let H be the subset of $M_2(\mathbb{R})$ consisting of all matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ for $a, b \in \mathbb{R}$. Is H closed under matrix multiplication? If so, give a (general) proof. If not, a specific example showing violation of closure is sufficient.

I.2.28. Prove or find a counterexample to the claim that every commutative binary operation on a set having two elements is associative. HINT: Consider all possible tables of such binary structures.

I.2.37. Suppose that $*$ is an associative and commutative binary operation on a set S. Show that $H = \{a \in S \mid a * a = a\}$ is closed under $*$. (The elements of H are *idempotents* of the binary operation $*$.)

I.3.8. Determine whether $\phi : M_2(\mathbb{R}) \to \mathbb{R}$ where $\phi(A)$ is the determinant of matrix A, is an isomorphism between $\langle M_2(\mathbb{R}), \cdot\rangle$ and $\langle \mathbb{R}, \cdot\rangle$. If so, then prove ϕ is an isomorphism and if not then explain which part of the definition of isomorphism is violated. Recall that an isomorphism is a one to one and onto mapping with the homomorphism property. HINT: For all $A, B \in M_2(\mathbb{R})$ we have $\det(AB) = \det(A)\det(B)$.

I.3.27. Prove that if $\phi : S \to S'$ is an isomorphism of $\langle S, *\rangle$ with $\langle S', *'\rangle$ and $\psi : S' \to S''$ is an isomorphism of $\langle S', *'\rangle$ with $\langle S'', *''\rangle$, then the composite function $\psi \circ \phi$ is an isomorphism of $\langle S, *\rangle$ with $\langle S'', *''\rangle$. You may assume that $\psi \circ \phi$ is one to one and onto, and so you only need to show that the homomorphism property is satisfied.

BONUS: Prove that $\psi \circ \phi$ of Problem I.3.27 is one to one and onto.

I.3.29. Prove that the commutivity of a binary operation is a structural property of a binary algebraic structure. That is, prove that if $*$ is commutative in $\langle S, *\rangle$ and $\phi : \langle S, *\rangle \to \langle S', *'\rangle$ is an isomorphism, then $*'$ is commutative in $\langle S', *'\rangle$.