Introduction to Algebra, MATH 5127

Homework 4, Sections I.4 and I.5

Due Friday February 15, 2013 at 2:30

I.4.2. Determine whether the binary operation on $2\mathbb{Z} = \{2n \mid n \in \mathbb{Z}\}$ defined as $a * b = a + b$ is a group or not. Check each of $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ and verify or find a counterexample to each.

I.4.19b. Let S be the set of all real numbers except -1, $S = \mathbb{R} \setminus \{-1\}$. Define $*$ on S by $a * b = a + b + ab$. Then $*$ is a binary operation on S. Prove that $\langle S, * \rangle$ is a group.

I.4.32. Show that every group G with identity e and such that $x * x = e$ for all $x \in G$ is abelian. HINT: Consider $(a * b) * (a * b)$.

I.4.34. (BONUS) Let G be a group with a finite number of elements. Show that for any $g \in G$, there exists an $n \in \mathbb{Z}^+ = \mathbb{N}$ such that $a^n = e$. See Exercise 33 for the meaning of a^n. HINT: Consider $e, a, a^2, a^3, \ldots, a^m$, where m is the number of elements in G, and use the cancellation laws.

I.4.41. Let G be a group and let g be one fixed element of G. Prove that the map i_g defined as $i_g(x) = gxg'$ for all $x \in G$ (where g' is the inverse of element g), is an isomorphism of G with itself. Notice that the notation used here implies that G is a "multiplicative group."

I.5.28. Find the order of the cyclic subgroup of group V generated by element c where the group table for V is given in Table 5.11 on page 51. Show your work!

I.5.41. Let $\phi : G \to G'$ be an isomorphism of a group $\langle G, * \rangle$ with a group $\langle G', *' \rangle$. Prove that if H is a subgroup of G, then $\phi[H] = \{\phi(h) \mid h \in H\}$ is a subgroup of G'. That is, an isomorphism carries subgroups into subgroups.

I.5.52. Let S be a subset of a group G. Define $H_S = \{x \in G \mid xs = sx \text{ for all } s \in S\}$. Prove that H_S is a subgroup of G. This subgroup is denoted H_G and is called the *center* of group G. Prove that H_G is abelian.