Introduction to Algebra, MATH 5127

Homework 8, Sections II.11 and III.13

Due Friday March 29, 2013 at 2:30

11.20. Are the groups $\mathbb{Z}_4 \times \mathbb{Z}_{18} \times \mathbb{Z}_{15}$ and $\mathbb{Z}_3 \times \mathbb{Z}_{36} \times \mathbb{Z}_{10}$ isomorphic? Give a detailed explanation and quote the results (by number) from this section. HINT: Write both groups as a product of indecomposable groups.

11.26. How many abelian groups (up to isomorphism) are there of order 24? of order 25? of order (24)(25)? Give a list of all nonisomorphic groups of each order.

11.39. Let G be an abelian group. Prove that the elements of finite order in G form a subgroup of G. This subgroup is called the *torsion subgroup* of G. HINT: If $g \in G$ is of order n, then $a^n = e$ where e is the identity of G and n is the smallest positive power for which $a^n = e$.

Note. If $G_1, G_2, \ldots, G_n$ are groups, then we have considered the direct product $\prod_{i=1}^n G_i$. It is not entirely correct to say that G_i is a subgroup of $\prod_{i=1}^n G_i$ (since, for example, the elements of $\prod_{i=1}^n G_i$ are n-tuples and not simply elements of the G_i). Strictly speaking,

$$\overline{G_i} = \{(e_1, e_2, \ldots, e_{i-1}, a_i, e_{i+1}, \ldots, e_n) \mid a_i \in G_i \text{ and } e_j \text{ is the identity in } G_j\}$$

is a subgroup of $\prod_{i=1}^n G_i$ and $\overline{G_i} \cong G_i$. For this reason, $\prod_{i=1}^n G_i$ is the *internal direct product* of the $\overline{G_i}$'s and the *external direct product* of the G_i's. This is really mostly a matter of notation, and we will often abbreviate all this to the statement "G_i is a subgroup of $\prod_{i=1}^n G_i$."

11.50. Let H and K be groups and let $G = H \times K$. With the notation above, show that (a) every element g of G is of the form $g = \overline{h}\,\overline{k}$ where $\overline{h} \in \overline{H}$ and $\overline{k} \in \overline{K}$, (b) $\overline{h}\,\overline{k} = \overline{k}\,\overline{h}$ for all $\overline{h} \in \overline{H}$ and $\overline{k} \in \overline{K}$, and (c) $\overline{H} \cap \overline{K} = \{e_G\}$ where e_G is the identity of G. HINT: Let e_H be the identity of H and e_K be the identity of K. Then $\overline{H} = \{(h, e_K) \mid h \in H\}$ and $\overline{K} = \{(e_H, k) \mid k \in K\}$.

11.51. Let H and K be subgroups of a group G such that: (1) for all $g \in G$, we have $g = hk$ for some $h \in H$ and $k \in K$, (2) $hk = kh$ for all $h \in H$ and $k \in K$, and (3) $H \cap K = \{e\}$ where e is the identity of G.

(a) Prove that for each $g \in G$, the expression $g = hk$ for $h \in H$ and $k \in K$ is unique.

(b) Prove that G is isomorphic to $H \times K$. Give the specific isomorphism ϕ and prove that it is an isomorphism.

13.7 Let $\phi_i : G_i \to G_1 \times G_2 \times \cdots \times G_i \times \cdots \times G_r$ be given by $\phi_i(g_i) = (e_1, e_2, \ldots, e_{i-1}, g_i, e_{i+1}, \ldots, e_r)$ where $g_i \in G_i$ and e_j is the identity in G_j. This is an *injection map*. Is ϕ a homomorphism?

13.29. Let G be a group and let $g \in G$. Let $i_g : G \to G$ be defined by $i_g(x) = gxg^{-1}$ for $x \in G$. Prove that i_g is an isomorphism of G with itself, called an *automorphism* of G.

13.51 Let G be any group and let $a \in G$. Let $\phi : \mathbb{Z} \to G$ be defined by $\phi(n) = a^n$. (a) Prove that ϕ is a homomorphism. (b) Describe the image $\phi[\mathbb{Z}]$. (c) Describe the possibilities for the kernel of ϕ.

13.52 Let $\phi : G \to G'$ be a homomorphism with kernel H and let $a \in G$. Prove that $\{x \in G \mid \phi(x) = \phi(a)\} = Ha$. NOTE: This result relates cosets to kernels of homomorphisms. This will be very important in Section 14. HINT: Let $A = \{x \in G \mid \phi(x) = \phi(a)\}$ and show that $A \subseteq Ha$ and $Ha \subseteq A$.