Introduction to Algebra, MATH 5127

Homework 9, Sections III.14 and III.15

Due Friday April 5, 2013 at 2:30

14.4. Consider the factor group $(\mathbb{Z}_3 \times \mathbb{Z}_5)/(\{0\} \times \mathbb{Z}_5)$. Find the order and elements of this group. List the elements of each coset of $\{0\} \times \mathbb{Z}_5$.

14.24. Prove that A_n, $n \geq 2$, is a normal subgroup of S_n (that is, show that the left cosets and right cosets of A_n are the same). Find the index $(S_n : A_n)$. To which familiar group is S_n/A_n isomorphic? HINT: The cosets of A_n consist of the set of all even permutations in S_n and the set of all odd permutations in S_n.

14.31. Suppose H and K are normal subgroups of G. That is, $H \triangleleft G$ and $K \triangleleft G$. Prove that $H \cap K$ is a normal subgroup of G. You may assume that $H \cap K$ is a subgroup of G (which is Exercise 5.54). HINT: Use Theorem 14.13 (twice, maybe three times).

14.34. Prove that if finite group G has exactly one subgroup H of a given order, then H is a normal subgroup. HINT: Apply the inner automorphism i_g to H. Use Exercise 5.41.

15.3. Consider the factor group $(\mathbb{Z}_2 \times \mathbb{Z}_4)/\langle(1,2)\rangle$. Find the order and elements of this group. List all elements of each coset of $\langle(1,2)\rangle$. Classify the factor group using the Fundamental Theorem of Finitely Generated Abelian Groups.

15.37. (Test 2, Number 5.) Let $Z(G)$ be the center of group G. Prove that if $G/Z(G)$ is cyclic then G is abelian. HINT: Since $G/Z(G)$ is the group of cosets of $Z(G)$, for $G/Z(G)$ cyclic there is some coset of $Z(G)$ which generates $G/Z(G)$, say $aZ(G)$. Then the cosets of $Z(G)$ (i.e., the elements of $G/Z(G)$) are $a^n Z(G)$ where $n \in \mathbb{Z}$.