Supplement. The Cayley-Dickson Construction and Nonassociative Algebras

Note. In this supplement, we explore the Cayley-Dickson construction which allows us to create a $2n$-dimensional algebra (over $\mathbb{R}$) from an n-dimensional algebra (over $\mathbb{R}$). We freely use the term "algebra" in discussing the small dimensional spaces, and formally define it later. By starting with $\mathbb{R}$, we can apply the construction to create $\mathbb{C}$, then apply the construction to $\mathbb{C}$ to create the quaternions $\mathbb{H}$, and apply the construction to $\mathbb{H}$ to create the octonions $\mathbb{O}$. The process can continue, but as we iterate the Cayley-Dickson construction we find that we loose more and more "structure." These notes are largely based on John C. Baez's "The Octonions," *Bulletin of the American Mathematical Society*, **39**(2), 145–205 (2002). A copy of the original paper is online in PDF. You can also view a webpage version on Baez's website.

Note. Sir William Rowan Hamilton (August 4, 1805–September 2, 1865) was an Irish mathematician, astronomer, and physicist. He was a professor of Astronomy at Trinity College Dublin, and a director at Dunsink Observatory near Dublin. Hamilton is known in the scientific world for his work in optics and classical mechanics; in particular, his reformulation of Newtonian mechanics into what today is called Hamiltonian mechanics. In mathematics, he is primarily known for his introduction of the quaternions, denoted $\mathbb{H}$ (in commemoration of Hamilton).

William Rowan Hamilton (image from the MacTutor History of Mathematics Archive biography of Hamilton, accessed 10/26/2022.

In 1833 (on November 4), Hamilton presented a paper to Royal Irish Academy in which he expressed the complex numbers as ordered pairs of real numbers and defined addition and multiplication of the pairs in such a way that the order pair (a, b) represents the complex number $a + bi$. The paper was published as "Theory of Conjugate Functions, or Algebraic Couples; with a Preliminary and Elementary Essay on Algebra as the Science of Pure Time," *The Transactions of the Royal Irish Academy*, **17**, 293–423 (1831). The paper can be viewed and downloaded from JSTOR (accessed 10/26/2022). (The dates of 1831 versus 1833 remain a mystery to your humble presenter!) On page 403 of the journal in "On the Addition, Subtraction, Multiplication, and Division, of Number-Couples, as combined with each other," Hamilton defines these arithmetic operations as (though it is unnecessary to define subtraction and division, since these are not really arithmetic operations but instead involve additive and multiplicative inverses):

$$(b_1, b_2) + (a_1, a_2) = (b_1 + a_1, b_2 + a_2); \quad (52.)$$

$$(b_1, b_2) - (a_1, a_2) = (b_1 - a_1, b_2 - a_2); \quad (53.)$$

$$(b_1, b_2)(a_1, a_2) = (b_1, b_2) \times (a_1, a_2) = (b_1 a_1 - b_2 + a_2, b_2 a_1 + b_1 a_2); \quad (54.)$$

$$\frac{(b_1, b_2)}{(a_1, a_2)} = \left(\frac{b_1 a_1 + b_2 a_2}{a_1^2 + a_2^2}, \frac{b_2 a_1 - b_1 a_2}{a_1^2 + a_2^2} \right) \quad (55.)$$

After several pages of computations with exponential and trigonometric functions (Hamilton is working on dealing with complex exponents in this), it is stated on pages 417 and 418 of the journal:

$$\sqrt{-1} = (0, 1) \quad (157.)$$

$$(a_1, a_2) = a_1 + a_2\sqrt{-1}, \quad (158.)$$

In this way, Hamilton ties his ordered pairs of real numbers (or "algebraic couples") and resulting equations involving two separate real equations to Cauchy's "(so called) Imaginary Equation[s]," as he states in a footnote on page 297 of the journal.

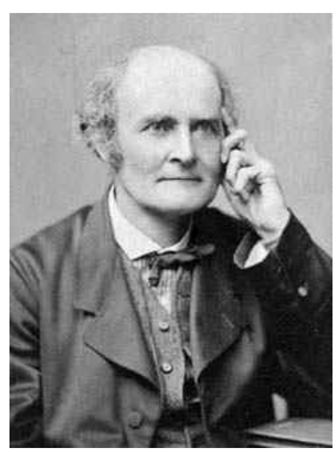
Note. In Complex Analysis 1 (MATH 5510), the complex field is defined using Hamilton's approach. See my online notes for Complex Analysis 1 on Section I.2. The Field of Complex Numbers. With $\mathbb{C} = \{(a, b) \mid a, b \in \mathbb{R}\}$, we define addition as $(a, b) + (c, d) = (a + c, b + d)$ and multiplication as $(a, b) \cdot (c, d) = (ac - bd, bc - ad)$. It is straightforward to confirm that $\mathbb{C}$ so defined is a field. We define the *conjugate* of $z = (a, b)$ and $\overline{z} = (a, b)^* = (a, -b)$ and the *modulus* of $z = (a, b)$ as $|z| = \sqrt{z\overline{z}} = \sqrt{a^2 + b^2}$ (we will use the term "norm" instead of modulus, in general). The ordered pair (a, b) is denoted $a + ib$, as expected.

Note. Sir William Rowan Hamilton introduced the quaternions to the mathematical world in the work "On Quaternions; or on a new System of Imaginaries in Algebra." This work appeared in 18 installments of *The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science* between 1844 and 1850. It appeared in volumes XXV to XXXVI. David R. Wilkins of the School of Mathematics, Trinity College, Dublin has a nice version of the work online (it runs 92 pages).

Note. The octonions were discovered in 1843 by John Graves, a friend of William R. Hamilton. Arthur Cayley independently discovered the octonions and published his result in "On Jacobi's Elliptic Functions, in reply to the Rev.; and on Quaternions," *Philosophical Magazine*, **26**(172), 208–211 (March, 1845); Cayley's work on the Quaternions appears as "P.S. *On Quaternions* on pages 210 and 211. Graves published his work in "On a Connection between the General Theory of Normal Couples and the Theory of Complete Quadratic Functions of Two Variables," *Philosophical Magazine*, **26**(173), 315-320 (April, 1845). So Graves work was first, but his publication appears very slightly after Cayley's.

John Thomas Graves

December 4, 1806–March 29, 1870



Arthur Cayley

August 16, 1821–January 26, 1895

These images are from the MacTutor History of Mathematics Archive biographies of Graves and Cayley (accessed 10/26/2022). The Thüringer Universistäts - und Landesbibliothek Jena website has copies of both Cayley's paper and Graves' paper which can be viewed online or downloaded (accessed 10/26/2022). Cayley's approach (in which he constructed the octonions by considering ordered pairs of quaternions) was later generalized by Leonard E. Dickson in "On Quaternions and Their Generalization and the History of the Eight Square Theorem," *Annals of Mathematics*, Second Series, **20**(3), 155-171 (1919). Dickson's paper is online on JSTOR (accessed 10/26/2022).

Leonard Dickson (January 22, 1874–January 17, 1954; image from the MacTutor History of Mathematics Archive biography of Dickson, accessed 10/26/2022.

The technique introduced by Cayley and generalized by Dickson, is now called the Cayley-Dickson Construction. We now describe the process as described in John C. Baez's "The Octonions," *Bulletin of the American Mathematical Society*, **39**(2), 145–205 (2001). We start with $\mathbb{R}$ and generate $\mathbb{C}$ as motivation.

Note. Notice that we can slightly modify Hamilton's definition of $\mathbb{C}$ in terms of ordered pairs of real numbers, as follows. We define $\mathbb{C} = \{(a,b) \mid a,b \in \mathbb{R}\}$ where we define addition as

$$(1') \qquad (a,b) + (c,d) = (a+c, b+d),$$

define multiplication as

$$(2) \qquad (a,b)(c,d) = (ac - db^*, a^*d + cb)$$

(where an asterisk indicates conjugation), and define conjugation of a complex number as

$$(3) \qquad (a,b)^* = (a^*, -b).$$

Since conjugation of a real number has no effect, then this is the same as Hamilton's definition of $\mathbb{C}$. We have introduced the unnecessary conjugation to establish a consistent pattern. First, multiplication in $\mathbb{C}$ is commutative and the multiplicative identity in $\mathbb{V}$ is $(1,0)$. Next, notice that

$$(a,b)(a,b)^* = (a,b)^*(a,b) = (a,b)(a,-b) = (a^2+b^2, -ab+ba)$$
$$= (a^2+b^2, 0) = (a^2+b^2)(1,0).$$

Hence, the (two sided) multiplicative inverse of nonzero (a,b) is

$$(a,b)^{-1} = \frac{1}{a^2+b^2}(a,-b) = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right).$$

This technique is the Cayley-Dickson Construction.

Note. We now apply the Cayley-Dickson Construction to create the quaternions from the complex numbers. We define the quaternions $\mathbb{H} = \{(a,b) \mid a,b \in \mathbb{C}\}$ where we define addition, multiplication, and conjugation as in $(1')$, (2), and (3):

$$(a,b)+(c,d) = (a+c, b+d), \quad (a,b)(c,d) = (ac-db^*, a^*d+cb), \quad (a,b)^* = (a^*, -b).$$

Notice that the conjugation is not inert this time, since in the order pairs it is applied to complex numbers. The multiplicative identity is still $(1,0)$ (notice that 0 and 1 are treated as complex numbers here), which we denote as $1 = (1,0)$. We define $i = (0,1)$, $j = (i,0)$, and $k = (0,i)$ (where i in the ordered pairs is the *complex* number i). With $c_1 = a + bi$ and $c_2 = c + di$ complex, we denote the quaternion (c_1, c_2) as

$$(c_1, c_2) = ((a,b),(c,d)) = a1 + bi + cj + dk \text{ where } a,b,c,d \in \mathbb{R}.$$

Notice that

$$i^2 = (0,1)^2 = ((0)(0) - (1)(1), (0)(1) + (0)(1)) = (-1,0) = -1,$$

$$j^2 = (i,0)^2 = ((i)(i) - (0)(0), (-i)(0) + (i)(0)) = (-1,0) = -1, \text{ and}$$

$$k^2 = (0,i)^2 = ((0)(0) - (i)(-i), (0)(i) + (0)(i)) = (-1,0) = -1.$$

We have that every real number $r \in \mathbb{R}$ commutes with i, j, and k: $ri = ir$, $rj = jr$, and $rk = kr$. Also,

$$ij = (0,1)(i,0) = ((0)(i) - (0)(1), (0)(0) + (i)(1)) = (0,i) = k \text{ and}$$

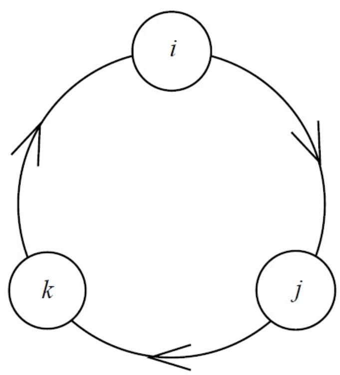
$$ji = (i,0)(0,1) = ((i)(0) - (1)(0), (-i)(1) + (0)(0)) = (0,-i) = -(0,i) = -k$$

so that $ij \neq ji$ (in fact, $ij = -ji = k$) and the quaternions are not commutative. We can similarly verify the familiar quaternionic identities $ijk = -1$, $jk = -kj = i$, and $ki = -ik = j$. These are the identifying equalities in the definition of the quaternions as a ring with additive abelian group $\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$; see my online notes for Modern Algebra 2 (MATH 5410) on Section III.1. Rings and Homomorphism. The inverse of nonzero $(a,b) \in \mathbb{H}$ is, as above,

$$(a,b)^{-1} = \frac{1}{a^2+b^2}(a,-b) = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$$

(see my online presentation The Quaternions: An Algebraic and Analytic Exploration for this computation). We can use the representation of a quaternion in terms of a linear combination of 1, i, j, and k to establish associativity of multiplication and the distribution laws. Therefore, $\mathbb{H}$ is a noncommutative division ring and an associative algebra of dimension 4 over $\mathbb{R}$ (a basis is $\{1, i, j, k\}$).

Note. The facts that $ij = k$, $jk = i$, $ki = j$, $ik = -j$, $kj = -i$, and $ji = -k$ are commonly illustrated as:



Multiplication of distinct elements of $\{i, j, k\}$ in a clockwise order produces $+1$ times the third element of the set, and multiplication of distinct elements of $\{i, j, k\}$ in a counterclockise produces -1 times the third element of the set. You are likely familiar with this behavior when considering cross-products of vectors in $\mathbb{R}^3$; see my online notes for Calculus 3 (MATH 2110) on Section 12.4. The Cross Product.

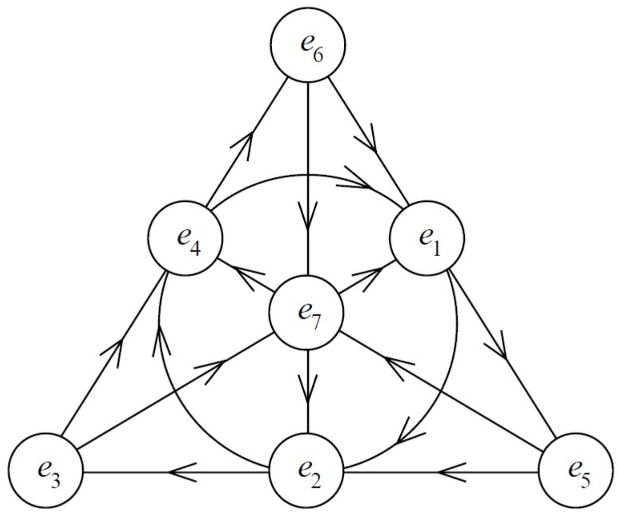
Note. Next we can apply the Cayley-Dickson Construction to create the octonions $\mathbb{O}$ from the quaternions. We define addition and multiplication of ordered pairs of quaternions in the same way as above. This gives a dimension 8 algebra over $\mathbb{R}$.

Similar to the treatment of the quaternions, we can take as a basis of the octonions $\{e_0 = 1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ so that for $a, b, c, d, e, f, g, h \in \mathbb{R}$ the octonion $\big(((a,b),(c,d)),((e,f),(g,h))\big)$ is denoted $a + be_1 + ce_2 + de_3 + ee_4 + fe_5 + ge_6 + he_7$. We get the following multiplication table for the e_i:

	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
e_4	e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
e_5	e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
e_6	e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	-1	e_2
e_7	e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1

We have $e_i^2 = -1$ for each $i \in \{1,2,3,4,5,6,7\}$, $e_ie_j = -e_je_i$ for $i \neq j$ (this is "anticommutivity"), $e_ie_j = e_k$ implies $e_{i+1}e_{j+1} = e_{k+1}$ (this is "index cycling"), and $e_ie_j = e_k$ implies $e_{2i}e_{2j} = e_{2k}$ (in this and the previous property, subscripts are reduced as appropriate; this is "index doubling"). We can use the multiplication table to show that multiplication is not associative: $(e_1e_2)e_3 = e_4e_3 = -e_6$ but $e_1(e_2e_3) = e_1e_5 = e_6$, so that $(e_1e_2)e_3 \neq e_1(e_2e_3)$.

Note. As with the quaternions, we can illustrate the multiplication table of the octonions with a diagram. The "Fano plane" is explored in Axiomatic and Transformational Geometry (MATH 5330) when covering finite geometry. See my online notes for Axiomatic and Transformational Geometry (MATH 5330) on Section 4.1. Projective Spaces. It also makes an appearance in Graph Theory 1 (MATH 5340) as an example of a geometric configuration; see my notes for that class on Section 1.3. Graphs Arising from Other Structures, and notice Figure 1.15(a). The following figure is from page 152 of Baez's paper.



This is a projective plane over the field $\mathbb{Z}_2$. It has seven lines, each containing three points (the points are indicated here by basis elements). The seven lines are the six collinear vertices in this picture, along with the line consisting of basis elements e_1, e_2, and e_4 (represented by the circle containing these). In fact, each line should be interpreted as a directed circle (as in the case of i, j, and k in the quaterions) where products taken in the direction of the orientation yields a "positive" output and products opposite to the direction of the orientation yields a "negative" output. For example (looking at the upper right line), $e_6e_1 = e_5$, $e_5e_6 = e_1$, $e_1e_6 = -e_5$, and $e_6e_5 = -e_1$. In fact, the basis elements along any one line of the Fano plane, along with basis elements $e_0 = 1$, generates a sub-algebra of $\mathbb{O}$ which is isomorphic to $\mathbb{H}$.

Note. Applying the Cayley-Dickson Construction to the octonions, we get a 16-dimensional algebra over $\mathbb{R}$, called the *sedenions* and denoted $\mathbb{S}$. Taking the basis as $\{e_0 = 1, e_1, e_2, \ldots, e_{15}\}$, we have the multiplication table:

	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6	e_9	$-e_8$	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	e_4	$-e_5$	e_{10}	e_{11}	$-e_8$	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$	e_{11}	$-e_{10}$	e_9	$-e_8$	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3	e_{12}	e_{13}	e_{14}	e_{15}	$-e_8$	$-e_9$	$-e_{10}$	$-e_{11}$
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	$-e_8$	e_{11}	$-e_{10}$
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	$-e_8$	e_9
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	$-e_8$
e_8	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_{12}$	$-e_{13}$	$-e_{14}$	$-e_{15}$	-1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_9	e_9	e_8	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$	$-e_1$	-1	$-e_3$	e_2	$-e_5$	e_4	e_7	$-e_6$
e_{10}	e_{10}	e_{11}	e_8	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}	$-e_2$	e_3	-1	$-e_1$	$-e_6$	$-e_7$	e_4	e_5
e_{11}	e_{11}	$-e_{10}$	e_9	e_8	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}	$-e_3$	$-e_2$	e_1	-1	$-e_7$	e_6	$-e_5$	e_4
e_{12}	e_{12}	e_{13}	e_{14}	e_{15}	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_4$	e_5	e_6	e_7	-1	$-e_1$	$-e_2$	$-e_3$
e_{13}	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	e_8	e_{11}	$-e_{10}$	$-e_5$	$-e_4$	e_7	e_6	e_1	-1	e_3	$-e_2$
e_{14}	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	e_8	e_9	$-e_6$	$-e_7$	$-e_4$	e_5	e_2	$-e_3$	-1	e_1
e_{15}	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	e_8	$-e_7$	e_6	$-e_5$	$-e_4$	e_3	e_2	$-e_1$	-1

Information seems harder to come by on the sedenions (the information presented here on the sedenions, including the multiplication table, is from the Wikipedia page on Sedenions). Since the quaternions are a subalgebra, then they are not commutative. Since the octonions are a subalgebra, then they are not commutative. In fact, we can use the table above to show that the sedenions even have zero divisors:

$$(e_1 + e_{10})(e_5 + e_{14}) = e_1e_5 + e_1e_{14} + e_{10}e_5 + e_{10}e_{14} = (-e_4) + e_{15} + (-e_{15}) + e_4 = 0.$$

A "division algebra," by definition, has no zero divisors so that the sedenions are not a division algebra. Surprisingly, we will see that multiplicative inverses are still present below.

Note/Definition. An algebra created by starting with $\mathbb{R}$ and iterating the Cayley-Dickson Construction is called a *Cayley-Dickson algebra* (as is the case on the Wikipedia page on the Cayley-Dickson Construction). We have (by induction) that, for any Cayley-Dickson algebra, the condition $(a,b)^*(a,b) = k(1,0)$ where $k \in \mathbb{R}$ and

Note. We now state some formal definitions and results from John C. Baez's "The Octonions," *Bulletin of the American Mathematical Society*, **39**(2), 145–205 (2001).

Definition. An *algebra* A is a vector space (over $\mathbb{R}$, in these notes) that is equipped with a bilinear map (that is, linear in both entries) $m : A \times A \to A$ called *multiplication* and a nonzero element $1 \in A$ called the *unit* such that $m(1,a) = m(a,1) = a$. We denote $m(a,b)$ as ab. An algebra A is a *division algebra* if for any $a,b \in A$ with $ab = 0$, we have either $a = 0$ or $b = 0$. A *normed division algebra* is an algebra A that is also a normed vector space with $\|ab\| = \|a\|\|b\|$.

Note. The bilinear comment in the definition of an algebra means that $m(a+b,c) = m(a,c) + m(b,c)$ and $m(a,b+c) = m(a,b) + m(a,c)$, which we denote in the more familiar $(a+b)c = ac+bc$ and $a(b+c) = ab+bc$. Also, if $\|ab\| = \|a\|\|b\|$ then $ab = 0$ implies either $a = 0$ or $b = 0$, justifying the name "normed division algebra."

Note. In these notes, **we do not assume that an algebra is associative under multiplication!!!** In Modern Algebra 2 (MATH 5420), an algebra is defined over a commutative ring with identity (which $\mathbb{R}$ certainly is) and required to be a ring, meaning that multiplication is associative under this definition. See my online notes for Modern Algebra 2 on Section IV.7. Algebras; notice Definition IV.7.1. In the literature, it is common to use the term "nonassociative algebra" to indicate that associativity is not assumed. The expression "not associative" is used to indicate an algebra in which associativity is known not to hold (see pages 1 and 2 of Richard Schafer's *An Introduction to Nonassociative Algebras*, NY: Academic Press, 1966). With the Cayley-Dickson construction we loose associativity fairly early on (with the construction of the octonions $\mathbb{O}$). Notice that we naturally have $\mathbb{R}$ as a substructure of an algebra since we can associate $\alpha \in \mathbb{R}$ with $\alpha 1 \in A$ where 1 is the unit (think of 1 as a vector here).

Definition. An algebra (not necessarily associative) has *multiplicative inverses* if for any nonzero $a \in A$ there is $a^{-1} \in A$ such that $aa^{-1} = a^{-1}a = 1$.

Note. Baez states the following without proof or reference. Your humble instructor has struggled to find a (reputable) reference!

Theorem CD.1. An associative algebra has multiplicative inverses if and only if it is a division algebra.

Note. The absence of associativity has some weird implications…at least for those of us used to its presence (such as in rings). As evidence, we have the following observation. Theorem CD.1 need not hold for algebras that are not associative. We'll see below that every Cayley-Dickson algebra has multiplicative inverses, but we already know that the the octonions have zero divisors, so the octonions have multiplicative inverses yet they are not a division algebra! Beware that our definition of a division *algebra* (in terms of zero divisors) is different from the definition of a division *ring* (for which multiplicative inverses are required).

Note. Now we more explicitly address conjugation, multiplicative inverses, and norms. Following Baez, we use the key ideas of a $*$-algebra and nicely normed.

Definition. A *conjugation* on an algebra A (not necessarily associative) is a real-valued linear map $* : A \to A$ satisfying $a^{**} = a$ and $(ab)^* = b^*a^*$ for all $a, b \in A$. A $*$-*algebra* is an algebra equipped with a conjugation. A $*$-algebra A is *real* if $a = a^*$ for all $a \in A$. The $*$-algebra is *nicely normed* if $a + a^* \in \mathbb{R}$ and $aa^* = a^*a > 0$ for all nonzero $a \in A$.

Definition. In a nicely normed $*$-algebra, as in $\mathbb{C}$, we can define *real part* and *imaginary part* of elements. We take

$$\mathrm{Re}(a) = \frac{a + a^*}{2} \in \mathbb{R} \text{ and } \mathrm{Im}(a) = \frac{a - a^*}{2}.$$

Define a *norm* on A by $\|a\| = \sqrt{aa^*}$.

Theorem CD.2. If A is a nicely normed $*$-algebra, then every nonzero element has a multiplicative inverse. Namely, for $a \in A$, $a \neq 0$, we have $a^{-1} = a^*/\|a\|^2$.

Definition. We consider three levels of associativity. An algebra is *power-associative* if the subalgebra generated by any one element is associative. It is *alternative* if the subalgebra generated by any two elements is associative. If the subalgebra generated by any three elements is associative, then the algebra is *associative*.

Note/Schafer's Theorem 3.1. Richard Schafer in his *An Introduction to Nonassociative Algebras*, states a theorem (Theorem 3.1, which he credits to Emil Artin) that an algebra A is alternative if and only if we have

$$(aa)b = a(ab), \qquad (ab)a = a(ba), \quad \text{and} \quad (ba)a = b(aa).$$

In fact, any two of these equations implies the remaining one, so it is common to take the first and last as the definition of alternative.

Theorem CD.3. If $*$-algebra A is nicely normed and alternative, then A is a normed division algebra.

Note. We now state five propositions from Baez and give proofs (which Baez declare as "...straightforward calculations; to prove them here would merely deprive the reader of the pleasure of doing so" see his page 155). Some of the proofs are lengthy because we show all details. The absence of commutativity is familiar, but the possible absence of associativity is strange and we deal with it with care.

Proposition 1. Starting with any $*$-algebra A, the new $*$-algebra A' that results from the Cayley-Dickson Construction is not real.

Proposition 2. $*$-algebra A is real (and thus commutative) if and only if Cayley-Dickson algebra A' constructed from A is commutative.

Proposition 3. $*$-algebra A is commutative and associative if and only if Cayley-Dickson algebra A' constructed from A is associative.

Proposition 4. $*$-algebra A is associative and nicely normed if and only if Cayley-Dickson algebra A' constructed from A is alternative and nicely normed.

Proposition 5. $*$-algebra A is nicely normed if and only if Cayley-Dickson algebra A' constructed from A is nicely normed.

Note. We now apply Propositions 1 through 5 to deduce properties of Cayley-Dickson $*$-algebras. First, we know that $\mathbb{R}$ is real, commutative, associative, and nicely normed (by absolute value), so that $\mathbb{C}$ is commutative (Proposition 2), associative (Proposition 3), and nicely normed (Proposition 5). Next, $\mathbb{H}$ is associative (Proposition 3) and nicely normed (Proposition 5). Then $\mathbb{O}$ is alternative and nicely normed (Proposition 4). Since $\mathbb{C}$ is not real, then $\mathbb{H}$ is not commutative by Proposition 2 (also, recall that $ij = k \neq ji = -k$). Since $\mathbb{H}$ is not commutative, then $\mathbb{O}$ is not associative by Proposition 3 (also, recall that $(e_1e_2)e_3 \neq e_1(e_2e_3)$). Of course, once a property (such as associativity) is lost by a Cayley-Dickson $*$-algebra, then it does not hold in any of the subsequent Cayley-Dickson $*$-algebras (since the earlier $*$-algebras are substructures of the later $*$-algebras). By Proposition 5, all Cayley-Dickson $*$-algebras are nicely normed. Since each of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are nicely normed and alternative, then they are normed division algebras by Theorem CD.C. Since $\mathbb{O}$ is not associative, then the sedenions $\mathbb{S}$ are not alternative and so may not be a division algebra. In fact, we saw above that the sedenions have zero divisors, $(e_1 + e_{10})(e_5 + e_{14}) = 0$, and so $\mathbb{S}$ is not a division algebra. However, all Cayley-Dickson $*$-algebras are nicely normed and so have multiplicative inverses by Theorem CD.2.

Note. As a closing comment, we mention that you may be familiar with a commutator in a ring: $[a, b] = ab - ba$. (This is not to be confused with the commutator in a group: $aba^{-1}b^{-1}$.) In a commutative ring, the commutator is always 0, so it is (when nonzero) a measure of the failure of commutivity in a noncommutative ring. You may also encounter this idea in quantum mechanics as applied to operators (see my online notes for Hilbert Spaces and Applications, at one time used in Applied Mathematics 1 [MATH 5610], on Section 7.3. Basic Concepts and Postulates of Quantum Mechanics). In fact, the Uncertainty Principle can be expressed in terms of the expectation value of a commutator of Hermitian operators (see Theorem 7.4.1 in the Hilbert space notes). Since we see that a Cayley-Dickson algebra may not be associative, we can similarly define an *associator* as $[a, b, c] = (ab)c - a(bc)$ for a, b, c in the Cayley Dickson algebra. Similarly to the commutator, the associator is 0 when associativity holds and (when nonzero) is a measure of the failure of associativity in a nonassociative Cayley-Dickson algebra.

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