Chapter 7. Selected Topics

Section 7.3. A Theorem of Frobenius

Note. Ferdinand Frobenius received a doctorate from the University of Berlin in 1870 unter the direction of Karl Weierstrass. He worked in Zürich, Switzerland from 1875 to 1892 at the Eidgenössiche Polytechnikum. It was during this time (in 1877) that he presented the theorem of this section. Following the death of Leopold Kronecker in 1891, a position became available at the University of Berlin which Frobenius took. In algebra, Frobenius made contributions to the representation theory of groups and the character theory of groups. His contributions to other areas of math include the areas of differential equations, elliptoc and Jacobi functions, the theory of biquadratic forms, and the theory of surfaces.

Ferdinand Frobenius (October 26, 1849–August 3, 1917)

This image and these historical notes are from the MacTutor History of Mathematics Archive biography of Frobenius (accessed 10/21/2022).

Note. Herstein states two "important facts" about the complex numbers:

FACT 1. Every polynomial of degree n over the field of complex numbers has all its n roots in the field of complex numbers.

FACT 2. The only irreducible polynomials over the field of real numbers are of degree 1 or 2.

He also states the in the previous chapter in Section 6.10. The first important fact is shown in Complex Variables (MATH 4337/5337). See my online notes for Complex Variables on Section 4.53. Liouville's Theorem and the Fundamental Theorem of Algebra (Fact 1 is given in Theorem 4.53.2, The Fundamental Theorem of Algebra). The second important fact is shown in, for example, Precalculus 1 (Algebra) (MATH 1710). See my online notes for Precalculus 1 on Section 4.6. Complex Zeros; Fundamental Theorem of Algebra (see Theorem 4.6.C and the note following it).

Note. Herstein defines a "division ring" in his Section 3.2 as follows.

Definition. A *division ring* is a ring for which the nonzero elements (that is, the elements other than the additive identity) form a group under multiplication.

Note. The following definition of an "algebra" is given in Herstein's Section 6.1. Recall that Herstein refer's to what is traditionally simply called a "ring" as an "associative ring" (see Herstein's Section 3.1; it is traditional to require associativity of multiplication in a ring, so we use slightly different terminology from Herstein here).

Definition. A ring A is an *algebra* over field F if A is a vector space over F such that for all $a, b \in A$ and $\alpha \in F$, we have $\alpha(ab) = (\alpha a)b = a(\alpha b)$.

Note/Definition. Recall that the *center of a group* is the set of elements of the group that commute with all other elements of the group (for example, the identity element is in the center). Similarly, the *center of a ring* is the set of elements of the ring that commute under multiplication with all other elements of the ring.

Definition. A division algebra D (that is, a division ring that is an algebra) is *algebraic over a field* F is:

1. F is contained in the center of D, and
2. every $a \in D$ satisfies a nontrivial polynomial with coefficients in F.

Note. Problem 7.3.1 is:

> If the division ring D is finite-dimensional, as a vector space, over the field F which is contained in the center of D, prove that D is algebraic over F.

However, division ring D can be algebraic over field F, yet D may not be finite-dimensional over F. An example of this is to be given in Problem 7.3.2 (in which D is even a field). First, we consider division ring which is algebraic over $\mathbb{C}$. We will use this in our exploration of division rings algebraic over $\mathbb{R}$ in the Theorem of Forbenius (Theorem 7.3.1).

Note. The quaternions, $\mathbb{H}$, are the standard example of a noncommutative division ring. The quaternions can be defined by starting with the abelian group $\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$ and representing (a_0, a_1, a_2, a_3) as the formal sum $a_0 1 + a_1 i + a_2 j + a_3 k$. Addition is defined component-wise, and multiplication is defined by assuming distribution over the formal sum and the equations:

$ri = ir$, $rj = jr$, and $rk = kr$ for all $r \in \mathbb{R}$,

$i^2 = j^2 = k^2 = ijk = -1$, $ij = -ji = k$, $jk = -kj = i$, and $ki = -ik = j$.

Associativity of multiplication then follows. For more details, see my online notes for Modern Algebra 2 (MATH 5420) on Section III.1. Rings and Homomorphism; an example is given in which the quaternions are introduced.

Lemma 7.3.1. Let $\mathbb{C}$ be the field of complex numbers and suppose that the division ring D is algebraic over $\mathbb{C}$. Then $D = \mathbb{C}$.

Note. Similar to the proof of Lemma 7.3.1, we have the following concerning a division ring which is algebraic over $\mathbb{R}$.

Lemma 7.3.A. Let division ring D be algebraic over $\mathbb{R}$ and let the center of D contain a copy of $\mathbb{C}$. Then $D = \mathbb{C}$.

Theorem 7.3.1. **(Frobenius)** Let D be a division ring algebraic over field $F = \mathbb{R}$, the field of real numbers. Then D is isomorphic to one of: the field of real numbers $\mathbb{R}$, the field of complex numbers $\mathbb{C}$, or the division ring of real quaternions $\mathbb{H}$.

Note. These ideas are covered in Modern Algebra 2 (MATH 5410), time permitting, in Section IX.5, Algebras, and Section IX.6, Division Algebras. Frobenius' Theorem is given in Corollary IX.6.8, as a corollary to the a theorem of Noether and Skolem that involves simple left Artinian rings, K-algebra isomorphisms, and inner automorphisms.

Note. We now give a quick synopsis of the three (up to isomorphism) division rings which are algebraic over the real numbers. You know he real numbers from Analysis (MATH 4217/5217) as a complete ordered field (see my online notes for Analysis 1 on Section 1.2. Properties of the Real Numbers as an Ordered Field and Section 1.3. The Completeness Axiom). The real numbers form a continuum (this is given by the completeness), unlike the field of rational numbers (or the field of algebraic numbers). From an algebraic perspective, the real numbers lack algebraic closure (that is, there are polynomials in $\mathbb{R}[x]$ that do not have roots in $\mathbb{R}$). The real numbers have a norm given by absolute value. You know the complex numbers from Introduction to Algebra 2 (MATH 4137/5137) and Modern Algebra 2 (MATH 5420) as the extension field $\mathbb{R}[i]$ where $i^2 = -1$. In Complex Variables (MATH 4337/5337) we introduce the complex numbers as pairs of real numbers with addition and multiplication defined in a predictable way (see my online notes for Complex Variables on Section 1.1. Sum and Products). An equivalent (but more rigorous) approach is taken in Complex Analysis 1 (MATH 5510; see my online notes for Complex Analysis 1 on Section I.2. The Field of Complex Numbers). The set of complex numbers are defined as $\mathbb{C} = \{(a, b) \mid a, b \in \mathbb{R}\}$. The field operations are defined as $(a,b)+(c,d) = (a+c, b+d)$ and $(a,b)\cdot(c,d) = (ac-bd, bc+ad)$. We denote (a,b) as $a+ib$ where $i = (0,1)$ (we can easily show that $i^2 = (0,1)\cdot(0,1) = -1$. The conjugate of $z = a+ib$ is defined as $\overline{z} = a-ib$ and the modulus (or norm) of z is $|z| = \sqrt{z\cdot\overline{z}} = \sqrt{a^2+b^2}$. The complex numbers form an algebraically closed field by the Fundamental Theorem of Algebra (which is easily proved from the properties of analytic functions of a complex variable; see, for example, my online Complex Variables notes on Section 4.53. Liouville's Theorem and the Fundamental Theorem of Algebra). The complex numbers are *not* ordered (see my Complex Analysis 1 notes on Supplement. Ordering the Complex Numbers). From an algebraic point of view, the complex numbers are where it's at because of their algebraic closure! The real numbers are a subfield of the complex numbers.

Note. You probably first encounter the quaternions in Introduction to Modern Algebra (MATH 4127/5127) as group of order 8 (see my notes on Section I.7. Generating Sets and Cayley Digraphs. You also see them in Modern Algebra 2 (MATH 5420) Section III.1. Rings and Homomorphism, as mentioned above. But there they are defined in independently of the complex numbers. They are not commutative so they do not form a field, but they do form a division ring. The complex numbers are a "substructure" of the quaternions; we can say that $\mathbb{C}$ is a sub-division ring of $\mathbb{H}$. By going beyond the complex numbers we have lost structure (namely, we have lost commutativity). It is reasonable to wonder if we can go further and create additional numerical structures. We can, but as we go further we loose additional structure. For example, the next structure is the *octonions*, $\mathbb{O}$, which lack both commutativity and associativity. The octonions form a "nonassociative algebra." The technique by which these new structures are constructed is the Cayley-Dickson procedure. This is explored in a supplement to this section on The Cayley-Dickson Construction and Nonassociative Algebras.

Revised: 10/24/2022