Part I. Groups and Subgroups

Section I.1. Introduction and Examples

Note. In this section, we introduce the complex numbers and some important subsets of the complex numbers which form examples of a main topic of modern algebra: groups.

"Definition." We define the complex numbers as $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$ where i had the algebraic property that $i^2 = -1$. Addition and multiplication satisfies all the familiar properties from $\mathbb{R}$. For $z = a + ib$, we call a the *real part* of a and b the *imaginary part* of z, denoted $a = \mathrm{Re}(z)$ and $b = \mathrm{Im}(z)$.

Note. Once the real numbers are axiomatically defined (as is done in Analysis 1) then the complex numbers can be developed analytically using the definition above. On the other hand, the complex numbers can be defined algebraically by extending $\mathbb{R}$ algebraically by i, the topic of Part VI of our text. In fact, the **Fundamental Theorem of Algebra** states that $\mathbb{C}$ is "algebraically closed." This implies that an n-degree polynomial equation $c_n z^n + c_{n-1} z^{n-1} + \cdots + c_2 z^2 + c_1 z + c_0 = 0$ (where $c_k \in \mathbb{C}$ for $k = 0, 1, 2, \ldots, n$ and the variable is z) has n solutions (counting multiplicity).

Note. For $z = a + bi$ and $z_2 = c + di$, we have

$$z_1 z_2 = (a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i.$$

Definition. For $z = a + bi$, define the *modulus* or *absolute value* of z as $|z| = \sqrt{a^2 + b^2}$.

Exercise 1.41. Recall the power series

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.
\end{aligned}$$

Use these series to prove Euler's Formula $e^{i\theta} = \cos\theta + i \sin\theta$. You may assume that these three series converge absolutely for all complex numbers (and hence the series can be rearranged without changing their values).

Solution. We have

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \sum_{\substack{n=0 \\ n \equiv 0 (\text{mod } 4)}}^{\infty} \frac{(i\theta)^n}{n!} + \sum_{\substack{n=0 \\ n \equiv 1 (\text{mod } 4)}}^{\infty} \frac{(i\theta)^n}{n!} + \sum_{\substack{n=0 \\ n \equiv 2 (\text{mod } 4)}}^{\infty} \frac{(i\theta)^n}{n!} + \sum_{\substack{n=0 \\ n \equiv 3 (\text{mod } 4)}}^{\infty} \frac{(i\theta)^n}{n!}$$

since the series converges absolutely

$$= \sum_{\substack{n=0 \\ n \equiv 0 (\text{mod } 4)}}^{\infty} \frac{(\theta)^n}{n!} + \sum_{\substack{n=0 \\ n \equiv 1 (\text{mod } 4)}}^{\infty} i\frac{\theta^n}{n!} + \sum_{\substack{n=0 \\ n \equiv 2 (\text{mod } 4)}}^{\infty} \left(-\frac{\theta^n}{n!}\right) + \sum_{\substack{n=0 \\ n \equiv 3 (\text{mod } 4)}}^{\infty} \left(-i\frac{\theta^n}{n!}\right)$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{\theta^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} (-1)^{k+1} \frac{\theta^{2k+1}}{(2k+1)!} = \cos\theta + i \sin\theta.$$

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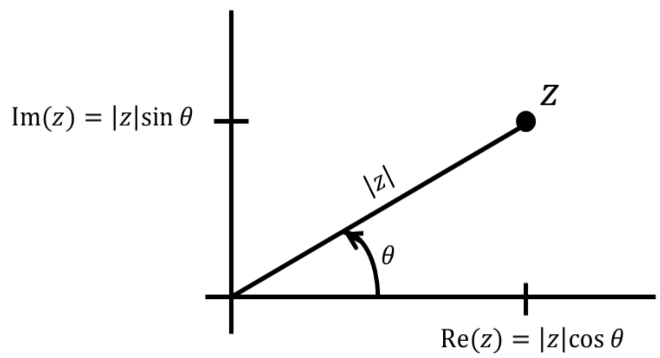
Notice. If $z = \cos\theta + i\sin\theta = e^{i\theta}$, then $|z| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$. We can then represent complex numbers in polar form as $z = |z|e^{i\theta} = |z|(\cos\theta + i\sin\theta)$:



θ is called an *argument* of z (notice that if θ is an argument of z then so is $\theta + 2k\pi$ for all $k \in \mathbb{Z}$).

Note. If $z_1 = |z_1|e^{i\theta_1}$ and $z_2 = |z_2|e^{i\theta_2}$, then

$$\begin{aligned}
z_1 z_2 &= \left(|z_1|e^{i\theta_1}\right)\left(|z_2|e^{i\theta_2}\right) \\
&= |z_1|(\cos\theta_1 + i\sin\theta_1)|z_2|(\cos\theta_2 + i\sin\theta_2) \\
&= |z_1||z_2|(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\
&= |z_1||z_2|(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2) \\
&= |z_1||z_2|\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right) \\
&\quad \text{since } \cos(a+b) = \cos a\cos b - \sin a\sin b \text{ and } \sin(a+b) = \cos a\sin b + \sin a\cos b \\
&= |z_1||z_2|e^{i(\theta_1+\theta_2)}.
\end{aligned}$$

So when multiplying complex numbers, we multiply moduli and add arguments.

Exercise 1.18. Solve the equation $z^3 = -8$.

Solution. Suppose $z = |z|e^{i\theta}$ and $z^3 = -8$. Then

$$z^3 = \left(|z|e^{i\theta}\right)^3 = |z|^3 e^{i(3\theta)} = |z|^3 \left(\cos(3\theta) + i\sin(3\theta)\right).$$

Since $|-8| = 8$, then $|z|^3 = 8$ and $|z| = 2$. Next, $-1 = \cos 3\theta + i \sin 3\theta$ and since an argument of -1 is of the form $\pi + 2k\pi$, we need $3\theta = \pi + 2k\pi$, or $\theta = \dfrac{\pi}{3} + \dfrac{2}{3}k\pi$ where $k \in \mathbb{Z}$. We get three non-coterminal values for θ: $\dfrac{\pi}{3}, \pi, \dfrac{5\pi}{3}$. This yields the three solutions:

$$\begin{aligned}
z_1 &= 2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right) = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 1 + \sqrt{3}i \\
z_2 &= 2(\cos\pi + i\sin\pi) = 2(-1 + 0i) = -2 \\
z_3 &= 2\left(\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right) = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 1 - \sqrt{3}i.
\end{aligned}$$

Note. We denote the set of all complex numbers of modulus 1 as U: $U = \{z \in \mathbb{C} \mid |z| = 1\}$. Since we multiply moduli when we multiply complex numbers, we see that a product of two elements of U is again in U—that is, U is *closed* under multiplication. In fact, the set U under multiplication is an example of a group.

Note. As in the previous example (where $n = 3$), we can compute the nth roots of unity $U_n = \{z \in \mathbb{C} \mid z^n = 1\}$. We find there are n such roots which can be computed as

$$\cos\left(m\frac{2\pi}{n}\right) + i\sin\left(m\frac{2\pi}{n}\right) \text{ for } m = 0, 1, 2, \ldots, n-1.$$

Since we add arguments when multiplying complex numbers, we find that U_n is closed under multiplication. In fact, the set U_n under multiplication is also an example of a group. In fact, since we add angles when multiplying, we find that the elements of U_n wrap around the unit circle $|z| = 1$ and multiplication of elements of U_n behave like the addition of hours on a clock. In fact, the structure of U_n is the same as that of $\mathbb{Z}_n$. Not surprisingly, addition modulo n is sometimes called "clock arithmetic."

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