Section I.2. Binary Operations

Note. In this section, we deal abstractly with operations on pairs (thus the term “binary”) of elements of a set. You are familiar with this concept in the settings of addition, subtraction, multiplication, and (except for 0) division of numbers. Two numbers, such as 9 and 3, yield through these four operations, the numbers 12, 6, 27, and 3, respectively. Notice that taking the 9 first and the 3 second affects the result for subtraction and division. That is, order matters for these operations.

Definition. A binary operation $\ast$ on a set $S$ is a function mapping $S \times S$ into $S$. For each (ordered pair) $(a, b) \in S \times S$, we denote the element $\ast((a, b)) \in S$ as $a \ast b$.

Example. The easiest examples of binary operations are addition and multiplication on $\mathbb{R}$. We could also consider these operations on different sets, such as $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{C}$.

Note. As we’ll see, we don’t normally think of subtraction and division as binary operations, but instead we think of them in terms of manipulation of inverse elements with respect to addition and multiplication (respectively).

Example. A more exotic example of a binary operation is matrix multiplication on the set of all $2 \times 2$ matrices. Notice that “order matters” (and there is, in general, no such thing as “division” here).
Definition 2.4. Let $*$ be a binary operation on set $S$ and let $H \subseteq S$. Then $H$ is closed under $*$ if for all $a, b \in H$, we also have $a * b \in H$. In this case, the binary operation on $H$ given by restricting $*$ to $H$ is the induced operation of $*$ on $H$.

Example. Let $\mathcal{E} = \{n \in \mathbb{Z} \mid n \text{ is even}\}$ and let $\mathcal{O} = \{n \in \mathbb{Z} \mid n \text{ is odd}\}$. Then, $\mathcal{E}$ is closed under addition (and multiplication). However, $\mathcal{O}$ is NOT closed under addition (but is closed under multiplication).

Example. Consider the set of all $2 \times 2$ invertible matrices. The set is closed under matrix multiplication (recall $(AB)^{-1} = B^{-1}A^{-1}$), but not closed under matrix addition.

Definition 2.11. A binary operation $*$ on a set $S$ is commutative if $a * b = b * a$ for all $a, b \in S$.

Example. Matrix multiplication on the set of all $2 \times 2$ matrices is NOT commutative.

Exercise 2.8a. Define $*$ on $\mathbb{Q}$ as $a * b = ab + 1$. Is $*$ commutative (prove or find a counterexample)?
Definition 2.12. A binary operation $\ast$ on a set $S$ is associative if $(a\ast b)\ast c = a\ast (b\ast c)$ for all $a, b, c \in S$.

Exercise 2.8b. Define $\ast$ on $\mathbb{Q}$ as $a \ast b = ab + 1$. Is $\ast$ associative (prove or find a counterexample)?

Note. We will study several algebraic structures by simply producing the “multiplication table” for the structure. For example, if $S = \{a, b, c\}$ and we have:

\[
\begin{align*}
  a \ast a &= b & a \ast b &= c & a \ast c &= b \\
  b \ast a &= a & b \ast b &= c & b \ast c &= b \\
  c \ast a &= c & c \ast b &= b & c \ast c &= a,
\end{align*}
\]

then we represent this binary operation as:

\[
\begin{array}{ccc}
  * & a & b & c \\
  a & b & c & b \\
  b & a & c & b \\
  c & c & b & a
\end{array}
\]

Notice that we read this as

$(i$th entry on left) $\ast$ $(j$th entry on top) = (entry in the $i$th row and $j$th column).

Notice $a \ast b = c$ and $b \ast a = a$, so $\ast$ is not commutative.

Notice. Binary operation $\ast$ is commutative if and only if table entries of it are symmetric with respect to the diagonal running from the upper left to the lower right.
Note. When defining a binary operation \(*\) on a set \(S\), we must make sure that (see page 24):

1. Exactly one element of \(S\) is assigned to each possible ordered pair of elements of \(S\) (that is, \(*\) is defined on all of \(S\) and \(*\) is “well defined”).

2. For each ordered pair of elements of \(S\), the value assigned to it is again in \(S\) (that is, \(S\) is closed under \(*\)).

Example 2.21. Define \(a \ast b = a/b\) on \(\mathbb{Z}^+ = \mathbb{N} = \{n \in \mathbb{Z} \mid n > 0\}\). Then \(\mathbb{N}\) is not closed under \(*\) since, for example, \(1 \ast 2 = 1/2 \notin \mathbb{N}\).

Exercise 2.20. On \(\mathbb{N}\) define \(*\) by letting \(a \ast b = c\) where \(c\) is the smallest integer greater than both \(a\) and \(b\). Is this a binary operation on \(\mathbb{N}\)?

Exercise 2.26. Prove that if \(*\) is an associative and commutative binary operation on a set \(S\), then

\[(a \ast b) \ast (c \ast d) = [(d \ast c) \ast a] \ast b\]

for all \(a, b, c, d \in S\).

Exercise 2.36. Suppose \(*\) is associative on \(S\). Let

\[H = \{a \in S \mid a \ast x = x \ast a \text{ for all } x \in S\}\]

Prove that \(H\) is closed under \(*\).