Section I.2. Binary Operations

Note. In this section, we deal abstractly with operations on pairs (thus the term "binary") of elements of a set. You are familiar with this concept in the settings of addition, subtraction, multiplication, and (except for 0) division of numbers. Two numbers, such as 9 and 3, yield through these four operations, the numbers 12, 6, 27, and 3, respectively. Notice that taking the 9 first and the 3 second affects the result for subtraction and division. That is, <u>order</u> matters for these operations.

Definition. A *binary operation* $*$ on a set S is a function mapping $S \times S$ into S. For each (ordered pair) $(a, b) \in S \times S$, we denote the element $*((a, b)) \in S$ as $a * b$.

Example. The easiest examples of binary operations are addition and multiplication on $\mathbb{R}$. We could also consider these operations on different sets, such as $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{C}$.

Note. As we'll see, we don't normally think of subtraction and division as binary operations, but instead we think of them in terms of manipulation of inverse elements with respect to addition and multiplication (respectively).

Example. A more exotic example of a binary operation is matrix multiplication on the set of all 2×2 matrices. Notice that "order matters" (and there is, in general, no such thing as "division" here).

Definition 2.4. Let $*$ be a binary operation on set S and let $H \subseteq S$. Then H is *closed* under $*$ if for all $a, b \in H$, we also have $a * b \in H$. In this case, the binary operation on H given by restricting $*$ to H is the *induced operation* of $*$ on H.

Example. Let $\mathcal{E} = \{n \in \mathbb{Z} \mid n \text{ is even}\}$ and let $\mathcal{O} = \{n \in \mathbb{Z} \mid n \text{ is odd}\}$. Then, $\mathcal{E}$ is closed under addition (and multiplication). However, $\mathcal{O}$ is NOT closed under addition (but is closed under multiplication).

Example. Consider the set of all 2×2 invertible matrices. The set is closed under matrix multiplication (recall $(AB)^{-1} = B^{-1}A^{-1}$), but not closed under matrix addition.

Definition 2.11. A binary operation $*$ on a set S is *commutative* if $a * b = b * a$ for all $a, b \in S$.

Example. Matrix mulitplication on the set of all 2×2 matrices is NOT commutative.

Exercise 2.8a. Define $*$ on $\mathbb{Q}$ as $a * b = ab + 1$. Is $*$ commutative (prove or find a counterexample)?

Definition 2.12. A binary operation $*$ on a set S is *associative* if $(a*b)*c = a*(b*c)$ for all $a, b, c \in S$.

Exercise 2.8b. Define $*$ on $\mathbb{Q}$ as $a * b = ab + 1$. Is $*$ associative (prove or find a counterexample)?

Note. We will study several algebraic structures by simply producing the "multiplication table" for the structure. For example, if $S = \{a, b, c\}$ and we have:

$$\begin{array}{ccc} a * a = b & a * b = c & a * c = b \\ b * a = a & b * b = c & b * c = b \\ c * a = c & c * b = b & c * c = a, \end{array}$$

then we represent this binary operation as:

$*$	a	b	c
a	b	c	b
b	a	c	b
c	c	b	a

Notice that we read this as

$$(i\text{th entry on left}) * (j\text{th entry on top}) = (\text{entry in the } i\text{th row and } j\text{th column}).$$

Notice $a * b = c$ and $b * a = a$, so $*$ is not commutative.

Notice. Binary operation $*$ is commutative if and only if table entries of it are symmetric with respect to the diagonal running from the upper left to the lower right.

Note. When defining a binary operation $*$ on a set S, we must make sure that (see page 24):

1. Exactly one element of S is assigned to each possible ordered pair of elements of S (that is, $*$ is defined on all of S and $*$ is "well defined").

2. For each ordered pair of elements of S, the value assigned to it is again in S (that is, S is closed under $*$).

Example 2.21. Define $a * b = a/b$ on $\mathbb{Z}^+ = \mathbb{N} = \{n \in \mathbb{Z} \mid n > 0\}$. Then $\mathbb{N}$ is not closed under $*$ since, for example, $1 * 2 = 1/2 \notin \mathbb{N}$.

Exercise 2.20. On $\mathbb{N}$ define $*$ by letting $a * b = c$ where c is the smallest integer greater than both a and b. Is this a binary operation on $\mathbb{N}$?

Exercise 2.26. Prove that if $*$ is an associative and commutative binary operation on a set S, then

$$(a * b) * (c * d) = [(d * c) * a] * b$$

for all $a, b, c, d \in S$.

Exercise 2.36. Suppose $*$ is associative on S. Let

$$H = \{a \in S \mid a * x = x * a \text{ for all } x \in S\}.$$

Prove that H is closed under $*$.