Part III. Homomorphisms and Factor Groups

Section III.13. Homomorphisms

Note. So far our study of algebra has been a study of the structure of groups. By "structure" I mean such properties as abelian or nonabelian, the number of generators, the orders of subgroups, the types of subgroups, etc. The idea behind a homomorphism between two groups is that it is a mapping which preserves the binary operation (from which all "structure" follows), but may not be a one to one and onto mapping (and so it may lack the preservation of the "purely set theoretic" properties, as the text says).

Definition 13.1. A map φ of a group G into a group G' is a *homomorphism* if for all $a, b \in G$ we have $\varphi(ab) = \varphi(a)\varphi(b)$.

Note. We see that $\varphi : G \to G'$ is an isomorphism if it is a one to one and onto homomorphism.

Note. There is always a homomorphism between any two groups G and G'. If e' is the identity element of G', then $\varphi(g) = e'$ for all $g \in G$ is a homomorphism called the *trivial homomorphism*.

Example 13.2. Suppose $\varphi : G \to G'$ is a homomorphism and φ is onto G'. If G is abelian then G' is abelian. Notice that this shows how we can get structure preservation without necessarily having an isomorphism.

Example 13.8. Let $G = G_1 \times G_2 \times \cdots \times G_i \times \cdots \times G_n$ be a direct product of groups $G_1, G_2, \ldots, G_n$. Define the *projection map* $\pi_i : G \to G_i$ where $\pi_i((g_1, g_2, \ldots, g_i, \ldots, g_n)) = g_i$. Then π_i is a homomorphism.

Exercise 13.10. Let F be the *additive* group of all continuous functions mapping $\mathbb{R}$ into $\mathbb{R}$. Let $\mathbb{R}$ be the *additive* group of real numbers and let $\varphi : F \to \mathbb{R}$ be given by $\varphi(f) = \int_0^4 f(x)\,dx$. Then φ is a homomorphism.

Definition 13.11. Let φ be a mapping of a set X into a set Y, and let $A \subseteq X$ and $B \subseteq Y$. The *image* of A in Y under φ is $\varphi[A] = \{\varphi(a) \mid a \in A\}$. The set $\varphi[X]$ is the *range* of φ (notice that φ is defined on all of X and we can take X as the domain of φ). The *inverse image* of B in X is $\varphi^{-1}[B] = \{x \in X \mid \varphi(x) \in B\}$.

Theorem 13.12. Let φ be a homomorphism of a group G into group G'.

(1) If e is the identity in G, then $\varphi(e)$ is the identity element e' in G'.

(2) If $a \in G$ then $\varphi(a^{-1}) = (\varphi(a)^{)}-1$.

(3) If H is a subgroup of G, then $\varphi[H]$ is a subgroup of G'.

(4) If K' is a subgroup of G', then $\varphi^{-1}[K]$ is a subgroup of G.

Note. Theorem 13.12 shows that homomorphisms map identities to identities, inverses to inverses, and subgroups (back and forth) to subgroups.

Note. By Theorem 13.12 part (4), we know that for $K < G'$, where $K = \{e'\}$, we have $\varphi^{-1}[K] < G$. This subgroup $\varphi^{-1}[K]$ includes all elements of G mapped under φ to e'. You encountered a similar idea in linear algebra when considering an $m \times n$ matrix A as a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. $\mathbb{R}^n$ and $\mathbb{R}^m$ are groups (by Theorem 11.2) and multiplication on the left by A is a homomorphism from $\mathbb{R}^n$ into $\mathbb{R}^m$ since $A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w}$ for all $\vec{v}, \vec{w} \in \mathbb{R}^n$ (this is Example 13.5). The identity in $\mathbb{R}^m$ is $\vec{0}$ and the elements of $\mathbb{R}^n$ mapped to $\vec{0}$ form the *nullspace* of A. That is, the nullspace of A is $A^{-1}[\{\vec{0}\}]$ (here, we use the symbol A^{-1} in the sense of an inverse *image* of a set under a mapping, not in the sense of the inverse of matrix A which may or may not exist). Recall that the nullspace of A is related to the invertibility of matrix A (and so, indirectly at least, to whether the mapping $A : \mathbb{R}^n \to \mathbb{R}^m$ is one to one and onto—that is, whether it is an isomorphism). In short, the elements of G mapped to the identity of G' are important!

Definition 13.13. Let $\varphi : G \to G'$ be a homomorphism. The subgroup $\varphi^{-1}[\{e'\}] = \{x \in G \mid \varphi(x) = e'\}$ (where e' is the identity in G) is the *kernel* of φ, denoted $\text{Ker}(\varphi)$.

Exercise 13.18. Let $\varphi : \mathbb{Z} \to \mathbb{Z}_{10}$ be a homomorphism such that $\varphi(1) = 6$. Find $\text{Ker}(\varphi)$.

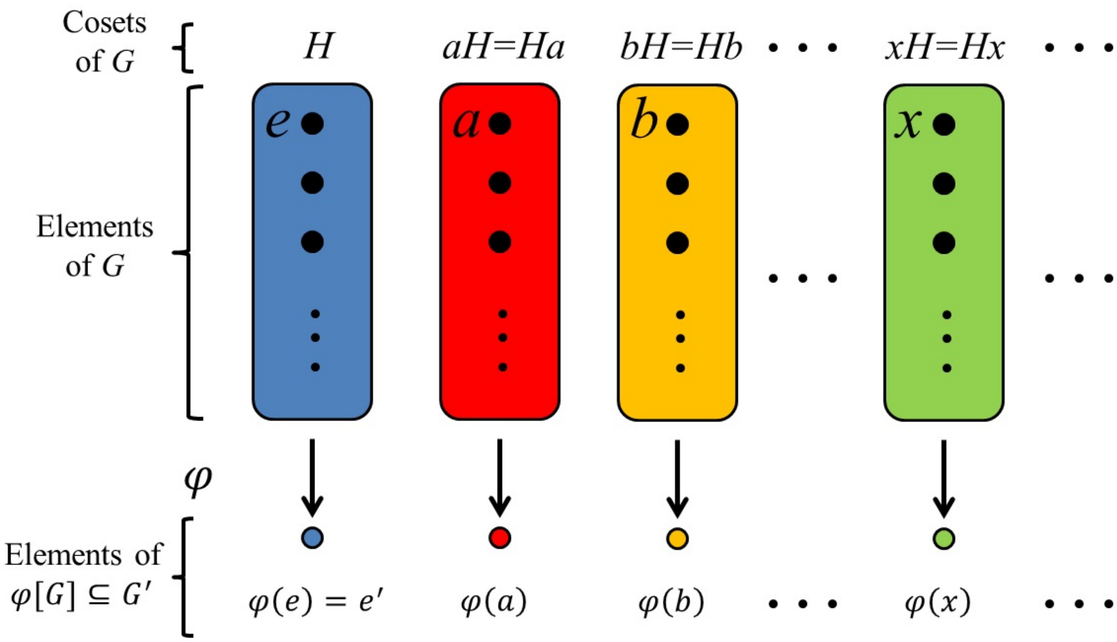
Note. The following result relates $\text{Ker}(\varphi)$ to cosets.

Theorem 13.15. Let $\varphi : G \to G'$ be a group homomorphism and let $H = \text{Ker}(\varphi)$. Let $a \in G$. Then the set

$$\varphi^{-1}[\{\varphi(a)\}] = \{x \in G \mid \varphi(x) = \varphi(a)\}$$

is the left coset aH of H, and is also the right coset Ha of H. So, the two partitions of G into left cosets and right cosets (see Section II.10) are the same.

Note. Recall that φ may not be one to one. In fact, if $\text{Ker}(\varphi) \neq \{e\}$, then we know that φ is not one to one (in fact this is an "if and only if" observation— see Corollary 13.18 below). So we can think of φ as a "many to one" mapping (for example, $f(x) = x^2$ is a two to one mapping for $x \neq 0$ since every nonzero element of the range is the image of two elements in the domain). Theorem 13.15 tells us that φ maps many elements of G onto single elements of G'. That is, φ maps the cosets aH and Ha onto the same element of G'—namely $\varphi(a)$. We exhibited a one to one mapping between the different cosets of G with respect to subgroup H in the Lemma stated before the proof of Lagrange's Theorem (see the class notes from Section II.10 or page 100 of the text), so all cosets of H are of the same cardinality (this cardinality is the "many" in the "many to one" mentioned above). We can think of φ as "collapsing down" (the text's wording—see page 129) the cosets of H onto individual elements of G'. See Figure 13.14 on page 130 or consider:



Notice, for example, that $\varphi^{-1}[\{\varphi(a)\}]$ is the coset $aH = Ha$. Now for the proof.

Example 13.17. Let D be the additive group of all differentiable functions mapping $\mathbb{R}$ into $\mathbb{R}$, and let F be the additive group of all functions mapping $\mathbb{R}$ into $\mathbb{R}$. Define $\varphi : D \to f$ as the differentiation operator $\varphi(f) = f'$. Then φ is a homomorphism since for all $f, g \in D$ we have

$$\varphi(f+g) = (f+g)' = f' + g' = \varphi(f) + \varphi(g).$$

Now $\text{Ker}(\varphi) = \{f \in D \mid f' = 0\}$, so $\text{Ker}(\varphi)$ is the set of all constant functions (which, of course, is a subgroup of D, call it C). We know $x^2 \in F$ so lets find all elements of D mapped under φ to x^2:

$$\{f \in D \mid \varphi(f) = f' = x^2 \in F\}.$$

We know from Calculus 1 (MATH 1910) that this is the *set* of all functions of the form $f(x) = \frac{1}{3}x^3 + k$ for some constant k. We denote this set as $\int x^2\,dx = \frac{1}{3}x^3 + C$. By Theorem 13.15, this set is the coset of $\text{Ker}(\varphi)$ of $x^2\,C$: $x^2\,C = \int x^2\,dx$.

Note. The following result tells us when a homomorphism φ is "one to one" versus "many to one," in terms of $\mathrm{Ker}(\varphi)$.

Corollary 13.18. A group homomorphism $\varphi : G \to G'$ is a one to one map if and only if $\mathrm{Ker}(\varphi) = \{e\}$.

Exercise 13.34. Is there a nontrivial homomorphism from $\mathbb{Z}_{12}$ to $\mathbb{Z}_4$?

Note. We'll see in the next section that when the left and right cosets of subgroup H are the same, $gH = Hg$ for all $g \in G$, then we can form a group out of the cosets of H (as discussed informally in Section II.10). Such subgroups H are useful in the study of nonabelian groups (we already have a classification of abelian groups in Theorem 11.12—at least, finitely generated abelian groups).

Definition 13.19. A subgroup H of a group G is *normal* if its left and right cosets coincide, that is if $gH = Hg$ for all $g \in G$. Fraleigh simply says "H is a normal subgroup of G," but a common notation is $H \triangleleft G$.

Note. If G is abelian, then all subgroups of G are normal.

Corollary 13.20. If $\varphi : G \to G'$ is a homomorphism, then $\mathrm{Ker}(\varphi)$ is a normal subgroup of G.

Note. We'll see more on normal subgroups in Section III.15 and a lot more in Section III.16 in connection with simple groups.

Exercise 13.50. Let $\varphi : G \to H$ be a group homomorphism. Then $\varphi[G]$ is abelian if and only if for all $x, y \in G$ we have $xyx^{-1}y^{-1} \in \mathrm{Ker}(\varphi)$.

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