

## Section III.15. Factor-Group Computations and Simple Groups

**Note.** In this section, we try to extract information about a group  $G$  by considering properties of the factor group (or “quotient group”)  $G/N$ . We also introduce a new class of groups, which have received much attention.

**Example.** Let  $G$  be a group with identity  $e$ . Consider the normal subgroup  $N = \{e\}$ . Since  $N$  has one element, then all cosets of  $N$  have one element—in fact, the coset  $gN = \{ge\} = \{g\}$ . So the factor group  $G/N$  is isomorphic to  $G$ .

**Example.** For any group  $G$ , the factor group  $G/G$  is isomorphic to  $\{e\}$ . This is because there is only one coset of  $G$  (treating  $G$  as a normal subgroup of  $G$ ) and so  $G/G$  is a group with one element.

**Note.** The above two examples are extreme cases of “collapse” of the cosets of  $G$  down to elements of  $G/N$ . If  $G$  is a finite group and  $N \neq \{e\}$  is a normal subgroup, then  $G/N$  is a smaller group than  $G$  and so “may have a more simple structure than  $G$ ” (using quotation marks when referring to the text’s wording). There is a correspondence between the multiplication of cosets in  $G/N$  and the multiplication of their representatives in  $G$  (as given in Corollary 14.5).

**Lemma.** If  $G$  is a finite group and  $N$  is a subgroup of  $G$  where  $|N| = |G|/2$ , then  $N$  is a normal subgroup of  $G$ .

**Example 15.4.** Consider  $S_n/A_n$ . Recall that the symmetric group  $S_n$  has order  $n!$  and the alternating group  $A_n$  has order  $n!/2$ . So, by Lemma,  $A_n$  is a normal subgroup and hence  $S_n/A_n$  is defined. So there are two cosets in  $S_n/A_n$ . As in the proof of Lemma, for any  $\sigma \in S_n$ , (1) if  $\sigma \in A_n$  (i.e.,  $\sigma$  is an even permutation) then  $\sigma A_n = A_n$ , and (2) if  $\sigma \in S_n \setminus A_n$  (i.e.,  $\sigma$  is an odd permutation) then  $\sigma A_n = S_n \setminus A_n$ . So the two cosets correspond to the even permutations and the odd permutations. If we denote cosets as  $E_n$  and  $O_n$ , we get the multiplication table for  $S_n/A_n$  as:

	$E_n$	$O_n$
$E_n$	$E_n$	$O_n$
$O_n$	$O_n$	$E_n$

As the text comments, the factor group  $S_n/A_n$  does not give us details about products of *specific elements* of  $S_n$ , but it does give us information about products of *types of elements* of  $S_n$  (even and odd, in this case).

**Example 15.6. Falsity of the Converse of the Theorem of Lagrange.**

We have claimed in the past that the alternating group  $A_4$  (of order  $4!/2 = 12$ ) does not have a subgroup of order 6. Recall that Lagrange's Theorem states that the order of a subgroup divides the order of its group. This example shows that there may be divisors of the order of a group which may not be the order of a subgroup.

**Exercise 15.6.** Classify  $\mathbb{Z} \times \mathbb{Z}/\langle(0, 1)\rangle$  according to the Fundamental Theorem of Finitely Generated Abelian Groups.

**Note.** The previous exercise foreshadows the following result.

**Theorem 15.8.** Let  $G = H \times K$  be the direct product of groups  $G$  and  $H$ . Then  $\overline{H} = \{(h, e) \mid h \in H\}$  is a normal subgroup of  $G$ . Also,  $G/\overline{H}$  is isomorphic to  $K$ . Similarly  $G/\overline{K}$  is isomorphic to  $H$  where  $\overline{K} = \{(e, k) \mid k \in K\}$ .

**Note.** The humble Theorem 15.8 reveals why we have been studying factor groups—because they allow us to FACTOR GROUPS (sometimes, at least)! It follows that if  $H$  is a normal subgroup of  $G = H \times K$  then  $G \cong (G/H) \times H$  since  $\overline{H} \cong H$ . So the reason we have addressed factor groups is so that we can take a given group and “factor” (or better yet, “decompose”) it into “smaller” groups. We know from the statements at the beginning of this section that  $H_1 = \{e\}$  and  $H_2 = G$  are normal subgroups of  $G$ , but they do not yield interesting factor groups because  $G/H_1 \cong G$  and  $G/H_2 \cong \{e\}$ . So very soon our attention will turn to normal subgroups which are neither trivial nor improper.

**Theorem 15.9.** A factor group of a cyclic group is cyclic.

**Exercise 15.4.** Classify  $(\mathbb{Z}_4 \times \mathbb{Z}_8)/\langle(1, 2)\rangle$ .

**Note.** We now introduce a new class of groups, which have a rich history. Details are given in the supplement to this section.

**Definition 15.14.** A group is *simple* if it is nontrivial and has no proper nontrivial normal subgroups.

**Example.** For any prime  $p$ , the group  $\mathbb{Z}_p$  is simple because it has no proper nontrivial subgroups by Lagrange's Theorem (and therefore has no proper nontrivial *normal* subgroups).

**Note.** We now see that simple groups play a role similar to that which prime numbers play in number theory! That is, if  $G$  is simple then the only factor groups of  $G$  are  $G/G \cong \{e\}$  and  $G/\{e\} \cong G$ . Notice that we cannot say that simple groups are equivalent to indecomposable groups. For example,  $\mathbb{Z}_{p^n}$ , where  $p$  is prime, is indecomposable but it is not simple when  $n > 1$ . We also cannot say that if  $H$  is a normal subgroup of  $G$ , then  $G \cong (G/H) \times H$  (as illustrated by considering indecomposable  $\mathbb{Z}_4$  and normal subgroup  $\mathbb{Z}_2 \cong \{0, 2\}$ ). The nature by which group  $G$  relates to its normal subgroups will be spelled out in more detail in the Jordan-Hölder Theorem (Theorem 35.15).

**Theorem 15.15.** The alternating group  $A_n$  is simple for  $n \geq 5$ .

**Note.** The proof we give of Theorem 15.15 is not elegant! It requires a clunky argument based on several cases. We defer it until the end of this section and give it as a supplement to these notes.

**Note.** We are interested in when  $G/N$  is simple for normal subgroup  $N$  of  $G$ . When this occurs, we have sort of found a maximal “factor” of  $G$ .

**Note.** Theorem 13.12 showed us that a group homomorphism sort of maps subgroups “back and forth.” The following result shows that the subgroup property of normality is also mapped back and forth by homomorphisms.

**Theorem 15.16.** Let  $\varphi : G \rightarrow G'$  be a group homomorphism. If  $N$  is a normal subgroup of  $G$ , then  $\varphi[N]$  is a normal subgroup of  $\varphi[G]$ . Also, if  $N'$  is a normal subgroup of  $\varphi[G]$ , then  $\varphi^{-1}[N']$  is a normal subgroup of  $G$ .

**Note.** The proofs of the two parts of Theorem 15.16 are Exercises 15.35 and 15.36.

**Definition 15.17.** A *maximal normal subgroup* of a group  $G$  is a normal subgroup  $M$  not equal to  $G$  such that there is no proper normal subgroup  $N$  of  $G$  properly containing  $M$ .

**Theorem 15.18.**  $M$  is a maximal normal subgroup of  $G$  if and only if  $G/M$  is simple.

**Note.** Personally, I find the proof in the text hard to follow (it may just be me, since the previous editions of Fraleigh present the same proof as given in the 7th edition). The proof depends on the use of the homomorphism of Theorem 14.9 which maps  $G$  to  $G/M$ . In addressing both “ $M$  is maximal” and “ $M/G$  is simple” we must be concerned with nontrivial and proper subgroups. Since a homomorphism is not one to one, this requires special attention to the details. We now present a proof with many more details than the proof given in Fraleigh (though still based on Fraleigh’s general argument’s). Notice that both claims in the theorem are established by contradiction. We now present [an alternative proof of Theorem 15.18](#).

**Definition.** For group  $G$ , define the *center* of  $G$  as

$$Z(G) = \{z \in G \mid zg = gz \text{ for all } g \in G\}.$$

**Note.** The letter  $Z$  in the notation above is from the German *zentrum* for “center.” In Exercise 5.52, it is shown that  $Z(G)$  is an abelian subgroup of  $G$ . Of course, if  $G$  itself is abelian, then  $Z(G) = G$ . Since  $Z(G)$  is abelian, then it is a normal subgroup of  $G$ .

**Definition.** For group  $G$ , consider the set

$$C = \{aba^{-1}b^{-1} \mid a, b \in G\}.$$

$C$  is the *commutator subgroup* of  $G$ . ( $C$  is shown to in fact be a group below.)

**Theorem 15.20.** Let  $G$  be a group. Then the set  $C = \{aba^{-1}b^{-1} \mid a, b \in G\}$  is a subgroup of  $G$ . Additionally,  $C$  is a normal subgroup of  $G$ . Furthermore, if  $N$  is a normal subgroup of  $G$  then  $G/N$  is abelian if and only if  $C \leq N$ .

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