Supplement. Finite Simple Groups

INTRODUCTION AND DEFINITIONS

Note. Recall that for H a subgroup of G, we define the *left cosets* of H as the sets $gH = \{gh \mid h \in H\}$, and the *right cosets* of H as the sets $Hg = \{hg \mid h \in H\}$, where $g \in G$ (Definition 10.2 of Fraleigh). All cosets of H are of the same size (page 100 of Fraleigh)—this fact was used in the proof of Lagrange's Theorem which states that the order of a subgroup is a divisor of the order of its group.

Note. Recall that N is a *normal subgroup* of group G, denoted $N \triangleleft G$, if the left cosets of N coincide with the right cosets of N: $gN = Ng$ for all $g \in G$ (Definition 13.19 of Fraleigh). Notice that for abelian groups, every subgroup is a normal subgroup.

Note. Recall that a group is *simple* if it is nontrivial and has no proper nontrivial normal subgroups (Definition 15.14 of Fraleigh).

Note. Since the cyclic groups $\mathbb{Z}_p$ for p prime have no proper nontrivial subgroups, then all of these groups are examples of simple groups. We have seen that all alternating groups A_n for $n \geq 5$ are simple (Theorem 15.15 of Fraleigh). This gives us two large classes of finite simple groups.

MOTIVATION: THE JORDAN-HÖLDER THEOREM

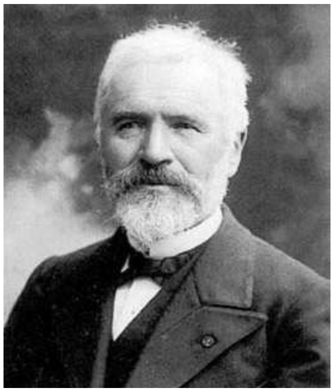
Note. Motivation for undertaking this classification project can be motivated by the Jordan-Hölder Theorem:

Jordan-Hölder Theorem. Every finite group G has a *composition series*

$$\{e\} = G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \cdots \triangleleft G_{n-1} \triangleleft G_n = G$$

where each group is normal in the next, and the series cannot be refined any further: in other words, each G_i/G_{i-1} is *simple* [Wilson, page 1].

This is equivalent to the statement of the theorem in Fraleigh (which appears in Section VII.35).



Camille Jordan (1838–1922)

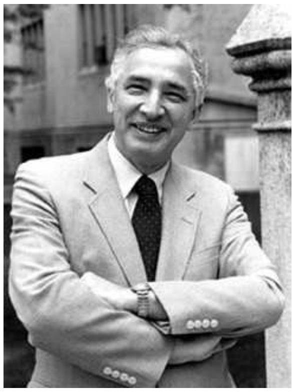
Otto Hölder (1859–1937)

Note. A common analogy is that creating a composition series of a group is similar to factoring a natural number into its prime factors. In a similar sense that all natural numbers are "built up" out of prime numbers, all finite groups are "built up" out of finite simple groups. Another analogy is that simple groups are like chemical elements out of which all molecules are made [Gallian].

THOUSANDS OF PAGES, HUNDREDS OF PAPERS

Note. Daniel Gorenstein in his 1982 *Finite Simple Groups: An Introduction to Their Classification* claims "In February 1981, the classification of the finite simple groups. . . was completed, representing one of the most remarkable achievements in the history of mathematics. Involving the combined efforts of several hundred mathematicians from around the world over a period of 30 years, the full proof covered something between 5,000 and 10,000 journal pages, spread over 300 to 500 individual papers" [Gorenstein, page 1].

Note. Not only is there an extraordinary number of research papers involved in the solution, but many of the papers themselves are quite long. One of the more impressive (and foundational) is a 255 page paper which filled an entire issue of the *Pacific Journal of Mathematics* in 1963. The paper, by Walter Feit and John Thompson is titled "Solvability of Groups of Odd Order" [*Pacif. J. Math.* **13**, 775–1029 (1963)] and contains a proof of a result which is very easy to state: Every finite group of odd order is solvable. Solvable groups are introduced in Section VII.35 of Fraleigh (and are related to solving polynomial equations).



Walter Feit (1930–2004)

John Thompson (1932–)

Note. Other lengthy papers of importance include [Gorenstein, page 3]:

1. John Thompson's classification of minimal simple groups (that is, simple groups in which all proper subgroups are solvable) appeared in six papers (totaling 410 journal pages) between 1968 and 1974 [*Bulletin of the American Mathematical Society* **74**, 383–437 (1968); *Pacific Journal of Mathematics* **33**, 451–536 (1970), **39**, 483–534 (1971), **48**, 511–592 (1973), **50**, 215–297 (1974), **51**, 573–630 (1974)].

2. John Walter's "The Characterization of Finite Groups with Abelian Sylow- 2-Subgroups" in a 109 page paper in 1969 [*Annals of Mathematics* **89**, 405–514 (1969)].

3. Alperin, Brauer, and Gorenstein's "Finite Groups with Quasi-Dihedral and Wreathed Sylow 2-Subgroups" *Transactions of the American Mathematical Society* **151**, 1–261 (1970).

4. Gorenstein and Harada's "Finite Groups whose 2-Subgroups are Generated by at most 4 Elements, *Memoir of the American Mathematical Society* **147**, 1–464 (1974).

STATEMENT OF THE CLASSIFICATION THEOREM

Note. The statement of the Classification Theorem for Finite Simple Groups is from *The Finite Simple Groups* by Robert A. Wilson, New York: Springer Verlag *Graduate Texts in Mathematics 251* (2009). The notation and format are Wilson's.

Theorem. The Classification Theorem for Finite Simple Groups.
Every simple group is isomorphic to one of the following:

(i) a cyclic group C_p of prime order p;

(ii) an alternating group A_n, for $n \geq 5$;

(iii) a classical group:

linear:	$\mathrm{PSL}_n(q)$, $n \geq 2$, except $\mathrm{PSL}_2(2)$ and $\mathrm{PSL}_2(3)$;
unitary:	$\mathrm{PSU}_n(q)$, $n \geq 3$, except $\mathrm{PSU}_3(2)$;
symplectic:	$\mathrm{PSp}_{2n}(q)$, $n \geq 2$, except $\mathrm{PSp}_4(2)$;
orthogonal:	$\mathrm{P}\Omega_{2n+1}(q)$, $n \geq 3$, q odd;
	$\mathrm{P}\Omega^+_{2n}(q)$, $n \geq 4$;
	$\mathrm{P}\Omega^-_{2n}(q)$, $n \geq 4$

where q is a power p^α of a prime p;

(iv) an exceptional group of Lie type:

$$G_2(q), q \geq 3; F_4(q); E_6(q);^2 E_6(q);^3 D_4(q); E_7(q); E_8(q)$$

where q is a prime power, or

$$^2B_2(2^{2n+1}), n \geq 1;^2 G_2(3^{2n+1}), n \geq 1;^2 F_4(2^{2n+1}), n \geq 1$$

or the Tits group $^2F_4(2)'$;

(v) one of 26 sporadic simple groups:

- the five Mathieu groups M_{11}, M_{12}, M_{22}, M_{23}, M_{24};
- the seven Leech lattice groups Co_1, Co_2, Co_3, McL, HS, Suz, J_2;
- the three Fischer groups Fi_{22}, Fi_{23}, Fi'_{24};
- the five Monstrous groups $\mathbb{M}$, $\mathbb{B}$, Th, HN, He;
- the six pariahs J_1, J_3, J_4, O'N, Ly, Ru.

Conversely, every group in this list is simple, and the only repetitions in this list are:

$$\begin{aligned} \mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5) &\cong A_5; \\ \mathrm{PSL}_2(7) &\cong \mathrm{PSL}_3(2); \\ \mathrm{PSL}_2(9) &\cong A_6; \\ \mathrm{PSL}_4(2) &\cong A_8; \\ \mathrm{PSU}_4(2) &\cong \mathrm{PSp}_4(3). \end{aligned}$$

Note. Both categories (iii) classical groups and (iv) exceptional groups of Lie type are examples of Lie groups. Lie groups are (usually) groups of continuous transformations. In 1874 Sophus Lie introduced his general theory of continuous transformation groups. Lie was trying to create a "Galois theory of differential equations" in which he could classify which differential equations could be solved by which classical techniques. He did not succeed in this goal, but his work lead to the successful formulation of such a theory by Charles Émile Picard and Ernest Vessoit [Kleiner, pages 21 and 22]. An easy example of a Lie group is the group of rotations of the unit circle in the xy-plane. This group is isomorphic to the multiplicative group U of all complex numbers of modulus 1: $\langle \{e^{i\theta} \mid \theta \in \mathbb{R}\}, \cdot \rangle$. A more sophisticated example is the multiplicative group of all quaternions of length 1: $\langle \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mid a^2 + b^2 + c^2 + d^2 = 1\}, \cdot \rangle$. This group is denoted $SU(2)$ and is closely related to a group of spatial rotations [Stillwell, page 1]. Notice that most of the groups above in categories (iii) and (iv) involve a parameter $q = p^\alpha$ where p is prime. This is because the continuous Lie groups can be "made discrete" by replacing the continuous variable with an element of a finite field. The classical groups then become matrices with entries from finite fields. Fields are introduced in Section IV.18 of Fraleigh.

THE SPORADIC GROUPS

Note. The first of the sporadic groups were discovered by Emile Mathieu in the 1860s, when he found five such groups (*Memoire sur le nombre de valeurs que peut acquerir une fonction quand on y permut ses variables de toutes les manières possibles*, *Crelle Journal* **5**, 9–42 (1860); *Memoure sur l'etude des fonctions de plusieures quantités, sur la mani'ere de les formes et sur les substitutions qui les lasissent invariables*, *Crelles Journal* **6**, 241–323 (1861); *Sur la fonction cinq fois transitive des 24 quantités*, *Crelles Journal* **18**, 25–46 (1873)]. In 1965, over a century after the previous discovery of a sporadic group, Zvonimir Janko found the group J_1 of order 175,560 [A New Finite Simple Group with Abelian 2-Sylow Subgroups and Its Characterization, *Journal of Algebra* **3**, 147–186 (1966)]. Over the next 10 years, another 20 sporadic groups were discovered, as listed below. The largest was the *monster group* of order approximate size 8.08×10^{53} "discovered" by Bernd Fischer and Robert Griess. The construction of the monster group was given by Robert Griess [The Friendly Giant, *Invent. Math.* **69**, 1–102 (1982); see Gorenstein page 2]. This rapid discovery of sporadic groups lead some to fear that there might be an infinite number of sporadic groups which would make the classification project intractable [Gorenstein, page 4].

Note. In more detail, the 26 sporadic groups and their orders are as follows (using the notation of [Ronan]):

Mathieu group $M11$	$2^4 \cdot 3^2 \cdot 5 \cdot 11 = 7,920$
Mathieu group $M12$	$2^6 \cdot 3^3 \cdot 5 \cdot 11 = 95,040$
Mathieu group $M22$	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 = 443,520$
Mathieu group $M23$	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23 = 10,200,960$
Mathieu group $M24$	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 = 244,823,040$
Janko group $J1$	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 = 175,560$
Janko group $J2$	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7 = 604,800$
Janko group $J3$	$2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19 = 50,232,960$
Janko group $J4$	$2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$ $= 86,775,571,046,077,562,880$
Conway group $Co1$	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23 = 4,157,776,806,543,360,000$
Conway group $Co2$	$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23 = 42,305,421,312,000$
Conway group $Co3$	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23 = 495,766,656,000$
Fischer group $Fi22$	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 = 64,561,751,654,400$
Fischer group $Fi23$	$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$ $= 4,089,470,473,293,004,800$
Fischer group $Fi24'$	$2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$ $= 1,255,205,709,190,661,721,292,800$
HigmanSims group HS	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11 = 44,352,000$
McLaughlin group McL	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 = 898,128,000$
Held group He	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17 = 4,030,387,200$
Rudvalis group Ru	$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29 = 145,926,144,000$
Suzuki group Suz	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 = 448,345,497,600$
O'Nan group ON	$2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31 = 460,815,505,920$
HaradaNorton group HN	$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19 = 273,030,912,000,000$
Lyons group Ly	$2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67 = 51,765,179,004,000,000$
Thompson group Th	$2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31 = 90,745,943,887,872,000$
Baby Monster group B	$2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$ $= 4,154,781,481,226,426,191,177,580,544,000,000$
Monster group M	$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47$ $\cdot 59 \cdot 71 = 808,017,424,794,512,875,886,459,$ $904,961,710,757,005,754,368,000,000,000$

HISTORY OF THE CLASSIFICATION PROJECT

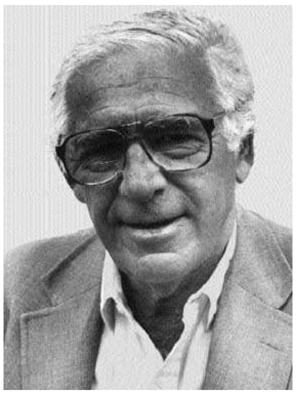
Note. Evariste Galois understood the importance of simple groups, since they arise in his study of polynomial equations, and Galois knew that the alternating groups A_n are simple for $n \geq 5$ (which we show in Fraleigh's Exercise 15.39). Nonetheless, finite group theory was slowly developed until the publication by Camille Jordan of *Traité des substitions des équations algébriques*, Gauthier-Villars (1870), and the Sylow Theorems in 1872 by the Norwegian Peter Sylow (1832–1918) [(1872), Théorèmes sur les groupes de substitutions, *Mathematische Annalen* **5**(4), 584-594 (1872)]. [See Wilson, page 1.] The Sylow Theorems are stated and proved in Section VII.36 of Fraleigh, and applications to the exploration of small simple groups are given in Section VII.37 (see page 331). A final noteworthy 19th century work, is William Burnside's book *Theory of Groups of Finite Order* which was first published in 1897 by Cambridge University Press. A second edition was published in 1911 and is still in print with Dover Publications and available through GoogleBooks (`http://books.google.com/books?id=rGMGAQAAIAAJ`, accessed 2/9/2013).

Peter Ludwig Sylow (1832–1918)

Leonard Dickson (1874–1954)

Note. Leonard Eugene Dickson (1874–1954), a prominent early American mathematician, worked on the classification of certain finite simple groups based on Lie algebras in the early 20th century [Wilson, page 2]. Richard Brauer started to study simple groups in the late 1940s and found a process (a relationship between the structure of a group and the "centralizers" of its "involutions") for addressing certain isomorphism results. He announced some of his results at the 1954 International Congress of Mathematicians in Amsterdam in 1954. His results were a starting point for the classification of simple groups in terms of centralizers of involutions. In 1955, Claude Chevalley published an influential paper on finite groups of Lie type [*Sur certains groupes simples*, *Tokoku Math. J.* **7**, 14–66 (1955)]. The lengthy paper of Feit and Thompson (mentioned above) appeared in 1963 and motivated the explosion of publications on finite simple groups which lead to the eventual classification theorem [Gorenstein, page 2].

Note. Daniel Gorenstein in a four lecture series outlined a 16 step program for classifying finite simple groups at a group theory conference at the University of Chicago in 1972. The program was published as an appendix to “The Classification of Finite Simple Groups: I,” *Bulletin of the American Mathematical Society, New Series* **1**, 43–199 (1979). The program was met with skepticism at the University of Chicago meeting, but another meeting in Duluth, Minnesota in 1976 involved presentations of theorems which indicated that the full classification theorem could be in sight [Gorenstein, page 5].



Daniel Gorenstein (1923–1992)

John H. Conway (1937–)

Note. Another prominent character in the classification project is John Horton Conway, currently (2013) of Princeton University (not to be confused with John B. Conway who was chair of the Department of Math at the University of Tennessee several years ago and who authored *Functions of One Complex Variable*, a Springer-Verlag book we often use in the graduate level Complex Analysis sequence [MATH 5510, 5520]). In the 1970s, Conway set out to create an atlas of the sporadic groups, called and denoted “The $\mathbb{ATLAS}$ Project.” At the time that the project started, new sporadic groups were still being discovered and constructed. The $\mathbb{ATLAS}$ was published by Oxford University Press in 1985 (reprinted with corrections and additions in November 2003) as the *Atlas of Finite Groups: Maximal Subgroups and Ordinary Characters for Simple Groups* by J. Conway, R. Curtis, S. Norton, R. Parker, and R. Wilson. It is an oversized spiral-bound book and the ETSU Sherrod Library owns a copy (QA171.A86 1985). Of course, the $\mathbb{ATLAS}$ does not list complete tables for the groups, but presents the structure in a simplified way.

Note. Gorenstein comments in his 1982 book that “. . . can one be certain that the ‘sieve’ has not let slip a configuration leading to yet another simple group? The prevailing opinion among finite group theorists is that the overall proof is accurate and that with so many individuals working on simple groups during the past fifteen years, often from such differing perspective, every significant configuration has loomed into view sufficiently often and could not have been overlooked” [Gorenstein, page

8]. However, there have been minor little problems in the proof discovered since the early 1980s. The Wikipedia webpage on the classification problem, `http://en.wikipedia.org/wiki/Classification_of_finite_simple_groups` (accessed 2/9/2013), lists several corrections to the original "proof" (of course, Wikipedia is not peer-reviewed and should not be thought of as authoritative).

Note. In contrast to the previous paragraph, notice that Wilson in his 2009 book states: "The project by Gorenstein, Lyons, and Solomon [D. Gorenstein, R. Lyons, R. Solomon, *The Classification of the Finite Simple Groups. Numbers 1 to 6*, American Mathematical Society, (1994, 1996, 1998, 1999, 2002, 2005)] to write down the whole proof in one place is still in progress: six of a projected eleven volumes have been published so far. The so-called 'quasithin' case is not included in this series, but has been dealt with in two volumes, totaling some 1200 pages, by Aschbacher and Smith [M. Aschbacher and S. D. Smith, *The Classification of Quasithin Groups I. Structure of Strongly Quasithin K-Groups*, American Mathematical Society (2004); M. Aschbacher and S. D. Smith, *The Classification of Quasithin Groups II. Main Theorems: The Classification of Simple QTKE-Groups*, American Mathematical Society (2004)]. Nor do they consider the problem of existence and uniqueness of the 26 sporadic simple groups: fortunately this is not in the slightest doubt. So by now most parts of the proof have been gone over by many people, and re-proved in different ways. Thus the likelihood of catastrophic errors is much reduced, though not completely eliminated" [Wilson, page 5].

Note. To put into perspective the enormity of this project, consider the most famous unsolved problem of the 20th century. Fermat's Last Theorem was unsolved for 350 years, until a proof was published by Andrew Wiles in 1995. The proof is presented in two papers which appeared in the *Annals of Mathematics* (the second paper includes a coauthor who helped with some errors which were detected in an earlier version of the proof) and the two papers combined take up 30 journal pages. More recently (around 2002), Grigori Perelman proved the 100 year old Poincare Conjecture in three papers which he posted online. The three papers together are 61 pages long.

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All photos are from The MacTutor History of Mathematics archive: `http://www-history.mcs.st-and.ac.uk/` (accessed 2/9/2013).

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