Supplement. Small Groups

Note. We have seen several examples of finite groups. The finite groups we have seen fall into the classes:

1. Cyclic Groups of order n (which are isomorphic to $\mathbb{Z}_n$),

2. Dihedral Groups D_n of order $2n$,

3. Symmetry Groups S_n or order $n!$,

4. Alternating Groups A_n or order $n!/2$, and

5. Direct Products of Groups.

Note. The information below can be found on the Wikipedia "List of Small Groups" webpage (accessed 7/13/2023). A more scholarly reference for these results is *The Small Group Library* compiled by Hans Ulrich Besche, Bettina Eick and Eamonn O'Brien. Information about which is on the The Small Groups Library website. The following table gives all groups of order 15 or less (up to isomorphism). There are 28 groups of order 15 or less, 20 of which are abelian. We have encountered all of the graphs in the table, except for the dicylic group of order 12 (our text does not address dicyclic groups, though the dicyclic group of order 8 is isomorphic to the quaternions).

Order	Group	Comments
1	$\mathbb{Z}_1$	The trivial group.
2	$\mathbb{Z}_2$	
3	$\mathbb{Z}_3 \cong A_3$	
4	$\mathbb{Z}_4$	
	Klein 4-group $V \cong \mathbb{Z}_2 \times \mathbb{Z}_2$	The smallest non-cyclic group.
5	$\mathbb{Z}_5$	
6	$\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$	
	$S_3 \cong D_3$	The smallest nonabelian group.
7	$\mathbb{Z}_7$	
8	$\mathbb{Z}_8$	
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	
	D_4	Nonabelian.
	Quaternions Q_8	Nonabelian.
9	$\mathbb{Z}_9$	
	$\mathbb{Z}_3 \times \mathbb{Z}_3$	
10	$\mathbb{Z}_{10} \cong \mathbb{Z}_2 \times \mathbb{Z}_5$	
	D_5	Nonabelian.
11	$\mathbb{Z}_{11}$	
12	$\mathbb{Z}_{12} \cong \mathbb{Z}_3 \times \mathbb{Z}_4$	
	$\mathbb{Z}_2 \times \mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$	
	$D_6 \cong \mathbb{Z}_2 \times D_3$	Nonabelian.
	A_4	Nonabelian; Smallest group which shows converse of Lagrange's Theorem does not hold.
	Dicyclic group of order 12	Nonabelian.
13	$\mathbb{Z}_{13}$	
14	$\mathbb{Z}_{14} \cong \mathbb{Z}_2 \times \mathbb{Z}_7$	
	D_7	Nonabelian.
15	$\mathbb{Z}_{15} \cong \mathbb{Z}_3 \times \mathbb{Z}_5$	

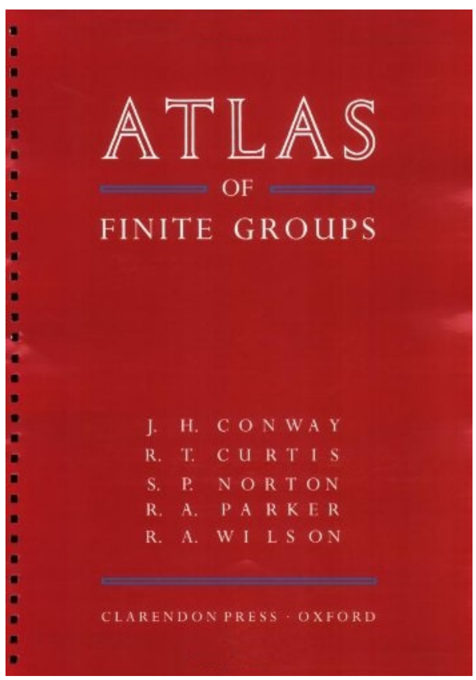
Note. When we consider groups of order 16, things become rather more complicated. There are the 5 abelian groups which can be found by the Fundamental Theorem of Finitely Generated Abelian Groups. There are 9 nonabelian groups of order 16. Three of the nonabelian groups are familiar: D_8, $\mathbb{Z}_2 \times D_4$, and $\mathbb{Z}_2 \times Q_8$. The other 6 nonabelian groups are fairly exotic (one of them is the dicyclic group of order 16). Details on the groups of order 16 can be found in "The Groups of Order Sixteen Made Easy" by Marcel Wild, *American Mathematical Monthly*, **112**, January 2005, 20–31. You can preview this paper on the JSTOR webpage, but you'll need your ETSU username and password to view the whole paper (accessed 7/13/2023).

Note. The lesson to be learned is that there is great diversity in groups! This fact makes it even more impressive that we can classify all finite abelian groups with the Fundamental Theorem of Finitely Generated Abelian Groups. One of the biggest results in the history of group theory is the classification of "finite simple groups" (we define *simple group* in Section III.15). This project involved decades of work (from the mid-1950s to the mid-1980s, with corrections trickling out as late as 2004) and hundreds of journal articles (see page 149 of Fraleigh). The final result states that finite simple groups *either* fall into three general categories (one is cyclic groups of prime order, another is alternating groups; the third category is harder to explain) *or* occur in a list of 26 "sporadic groups." The 26 sporadic groups have received a good deal of attention themselves. Some are truly gigantic groups—the largest one is called the Monster Group and is of order:

$$2^{46} \cdot 3^{30} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$$

$$= 808017424794512875886459904961710757005754368000000000 \approx 8.08 \times 10^{53}.$$

It might seem strange to mention the Monster Group in a list of "small groups," but it is finite and that's a lot smaller than any infinite group! As part of the classification project, the *Atlas of Finite Groups: Maximal Subgroups and Ordinary Characters for Simple Groups* by J. Conway, R. Curtis, S. Norton, R. Parker, and R. Wilson, was published by Oxford University Press in 1985 (reprinted with corrections and additions in November 2003). It is an oversized spiral-bound book which describes 129 individual groups (by my count).



The *Atlas of Finite Groups* is available in the Sherrod Library (QA171.A86 1985).

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