Summary Notes. For Section X.53

Note. The following results are a summary of the "Greatest Hits" of the first half of Section X.53.

Recall. If $F \leq E$, the collection of all automorphisms of E leaving F fixed forms a group $G(E/F)$.

Definition 51.7. A finite extension E of F is a *separable extension field of* F if $\{E : F\} = [E : F]$.

Definition 50.1. A field $E \leq \overline{F}$ is the *splitting field of* $\{f_i(x) \mid i \in I\}$ *over* F if E is the smallest subfield of $\overline{F}$ containing F and all the zeros in $\overline{F}$ of each of the $f_i(x)$ for $i \in I$. A field $K \leq \overline{F}$ is a *splitting field over* F if it is the splitting field of some set of polynomials in $F[x]$.

Definition 53.1. A finite extension K of F is a *finite normal extension of* F if K is a separable splitting field over F.

Note. K is a finite normal extension of F if and only if $|G(K/F)| = \{K : F\} = [K : F]$. (See the Note after Definition 53.)

Theorem 48.11. Let $\{\sigma_i \mid i \in I\}$ be a collection of automorphisms of a field E. Then the set $E_{\{\sigma_i\}}$ of all $a \in E$ fixed by every σ_i for $i \in I$ forms a subfield of E. (In particular, for F a subfield of K and $H < G(K/F)$, K_H is the subfield of K left fixed by all automorphisms of K in H.)

Note. For a field E, where $F \leq E \leq K$, let $\lambda(E)$ be the subgroup of $G(K/F)$ leaving E fixed.

Theorem 53.6. The Main Theorem of Galois Theory. (Partial)
Let K be a finite normal extension of a field F, with Galois group $G(K/F)$. For a field E, where $F \leq E \leq K$, let $\lambda(E)$ be the subgroup of $G(K/F)$ leaving E fixed. Then λ is a one to one map of the set of all such intermediate fields E onto the set of all subgroups of $G(K/F)$:

$$
\begin{tikzcd}
K \arrow[d, dash] \arrow[ddr] & G(K/F) \arrow[d, dash] \\
E \arrow[r, "\lambda"] \arrow[d, dash] & G(K/E) \arrow[d, dash] \\
F \arrow[uur] & \{\iota\}
\end{tikzcd}
$$

Revised: 4/13/2015