Note. In Part III we explored homomorphisms of groups and used the kernel of a homomorphism to create a group of cosets called the factor group (or quotient group) of the group modulo the kernel. In this section, we parallel this development but now for rings.

**Definition 26.1.** A map $\varphi$ of a ring $R$ into a ring $R'$ is a ring homomorphism if $\varphi(a + b) = \varphi(a) + \varphi(b)$ and $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in R$.

Note. As you have seen in the past, a homomorphism is a structure preserving map. For a ring, the “structure” consists of the operations of $+$ and $\cdot$.

**Example 26.2.** Let $R_1, R_2, \ldots, R_n$ be rings. For each $i$, $i = 1, 2, \ldots, n$, the map $\pi_i((r_1, r_2, \ldots, r_n)) = r_i$ is a homomorphism called the projection onto the $i$th component.

Note. The following several results on ring homomorphisms are analogous to the results developed in Section 13 for group homomorphisms.
Theorem 26.3. (Analogue of Theorem 13.12.)

Let \( \varphi \) be a homomorphism of a ring \( R \) into a ring \( R' \). If 0 is the additive identity in \( R \), then \( \varphi(0) = 0' \) is the additive identity in \( R' \), and if \( a \in R \), then \( \varphi(-a) = -\varphi(a) \).

If \( S \) is a subring of \( R \), then \( \varphi[S] \) is a subring of \( R' \). If \( S' \) is a subring of \( R' \), then \( \varphi^{-1}[S'] \) is a subring of \( R \). If \( R \) has unity 1, then \( \varphi(1) \) is unity for \( \varphi[R] \).

Definition 26.4. Let a map \( \varphi : R \to R' \) be a ring homomorphism. The subring

\[
\varphi^{-1}[0'] = \{ r \in R \mid \varphi(r) = 0' \}
\]

is the kernel of \( \varphi \), denoted Ker(\( \varphi \)).

Theorem 26.5. (Analogue of Theorem 13.15.)

Let \( \varphi : R \to R' \) be a ring homomorphism and let \( H = \text{Ker}(\varphi) \). Let \( a \in R \). Then \( \varphi^{-1}[\varphi(a)] = a + H = H + a \), where \( a + H = H + a \) is the coset containing \( a \) of the commutative additive group \( \langle H, + \rangle \).

Corollary 26.6. (Analogue of Theorem 13.18.)

A ring homomorphism \( \varphi : R \to R' \) is a one to one map if and only if \( \text{Ker}(\varphi) = \{0\} \).

Note. The following several results on factor rings are analogous to the results developed in Section 14 for factor groups.
Theorem 26.7. (Analogue of Theorem 14.1.)

Let \( \varphi : R \to R' \) be a ring homomorphism with kernel \( H \). Then the additive cosets of \( H \) form a ring \( R/H \) whose binary operations are defined by choosing representatives. That is, the sum of two cosets is defined by \((a + H) + (b + H) = (a + b) + H\) and the product of the cosets is defined by \((a + H)(b + H) = (ab) + H\). Also, the map \( \mu : R/H \to \varphi[R] \) defined by \( \mu(a + H) = \varphi(a) \) is an isomorphism.

Example 26.8. Consider \( \varphi : \mathbb{Z} \to \mathbb{Z}_n \) where \( \varphi(x) = x \mod n \). This is shown to be a ring homomorphism in Example 18.11 (page 171). By Theorem 26.7, we can compute sums and products in the ring \( \mathbb{Z}/n\mathbb{Z} \) (\( \text{Ker}(\varphi) = n\mathbb{Z} \)) using coset representatives. Also, by Theorem 26.7, we see that \( \mu : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}_n \) is an isomorphism.

Note. When we defined the factor group \( G/H \) we originally based it on the kernel \( H \) of a homomorphism. Following that, we defined a factor group \( G/H \) in terms of a normal subgroup \( H \). We now look for a condition on a subring of a ring, which will correspond to the condition on normal subgroups, so that we might generate factor rings in a setting other than that of homomorphisms.

Theorem 26.9. (Analogue of Theorem 14.4.)

Let \( H \) be a subring of the ring \( R \). Multiplication of additive cosets of \( H \) is well defined by the equation \((a + H)(b + H) = ab + H\) if and only if \( ah, hb \in H \) for all \( a, b \in R \) and \( h \in H \).
**Note.** Theorem 26.9 tells us that we can define multiplication of additive cosets if (and only if)

\[ aH = \{ah \mid h \in H\} \subseteq H \text{ and } Hb = \{hb \mid h \in H\} \subseteq H \]

for all \( a, b \in R \). This is the property of rings analogous to the property of normal subgroups which allowed us to produce factor groups. We’ll denote subgroups with this property with an \( N \).

**Definition 26.10.** An additive subgroup \( N \) of a ring \( R \) satisfying the properties \( aN \subseteq N \) and \( Nb \subseteq N \) for all \( a, b \in R \) is an **ideal**.

**Example 26.13.** Let \( F \) be the ring of all functions mapping \( \mathbb{R} \) into \( \mathbb{R} \). Let \( N \) be the subring of \( F \) of all functions \( f \) such that \( f(2) = 0 \). Then \( N \) is an ideal in \( F \). This is because for \( f \in F \) and \( g \in N \) we have

\[ fN = \{fg \mid g \in N\} = \{f(x)g(x) \mid g(2) = 0\} \subseteq N \]

since for \( f(x) \in F, f(2)g(2) = f(2)0 = 0 \). Similarly, \( Nf \subseteq N \).

**Corollary 26.14.** (*Analogue of Corollary 14.5.*)

Let \( N \) be an ideal of a ring \( R \). Then the additive cosets of \( N \) form a ring \( R/N \) with the binary operations defined by \((a + N) + (b + N) = (a + b) + N\) and \((a + N)(b + N) = ab + N\).
Definition 26.15. The ring \( R/N \) in Corollary 26.14 is the *factor ring* (or *quotient ring*) of \( R \) by \( N \).

**Note.** The following two results complete the analogies between factor groups (quotient groups) and factor rings (quotient rings).

**Theorem 26.16.** (Analogue of Theorem 14.9.)

Let \( N \) be an ideal of a ring \( R \). Then \( \gamma : R \to R/N \) given by \( \gamma(x) = x + N \) is a ring homomorphism with kernel \( N \).

**Theorem 26.17.** Fundamental Homomorphism Theorem (Analogue of Theorem 14.11.)

Let \( \varphi : R \to R' \) be a ring homomorphism with kernel \( N \). Then \( \varphi[R] \) is a ring and the map \( \mu : R/N \to \varphi[R] \) given by \( \mu(x + N) = \varphi(x) \) is an isomorphism. If \( \gamma : R \to R/N \) is the homomorphism given by \( \gamma(x) = x + N \) then for each \( x \in R \), we have \( \varphi(x) = (\mu\gamma)(x) \).

**Note.** So ideals for rings are analogous to normal subgroups for groups—they allow us to define quotient rings (respectively, quotient groups). The next exercise is further evidence of this.
Exercise 26.22. Let $\varphi : R \to R'$ be a ring homomorphism and let $N$ be an ideal of $R$.

(a) Then $\varphi[N]$ is an ideal of $\varphi[R]$.

(c) Let $N'$ be an ideal either of $\varphi[R]$ or of $R'$. Then $\varphi^{-1}[N']$ is an ideal of $R$.  

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