Part V. Ideals and Factor Rings

Section V.26. Homomorphisms and Factor Rings

Note. In Part III we explored homomorphisms of groups and used the kernel of a homomorphism to create a group of cosets called the factor group (or quotient group) of the group modulo the kernel. In this section, we parallel this development but now for rings.

Definition 26.1. A map φ of a ring R into a ring R' is a *ring homomorphism* if $\varphi(a+b) = \varphi(a) + \varphi(b)$ and $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in R$.

Note. As you have seen in the past, a homomorphism is a structure preserving map. For a ring, the "structure" consists of the operations of $+$ and $\cdot$.

Example 26.2. Let $R_1, R_2, \ldots, R_n$ be rings. For each i, $i = 1, 2, \ldots, n$, the map $\pi_i((r_1, r_2, \ldots, r_n)) = r_i$ is a homomorphism called the *projection onto the ith component*.

Note. The following several results on ring homomorphisms are analogous to the results developed in Section 13 for group homomorphisms.

Theorem 26.3. (Analogue of Theorem 13.12.)

Let φ be a homomorphism of a ring R into a ring R'. If 0 is the additive identity in R, then $\varphi(0) = 0'$ is the additive identity in R', and if $a \in R$, then $\varphi(-a) = -\varphi(a)$. If S is a subring of R, then $\varphi[S]$ is a subring of R'. If S' is a subring of R', then $\varphi^{-1}[S']$ is a subring of R. If R has unity 1, then $\varphi(1)$ is unity for $\varphi[R]$.

Definition 26.4. Let a map $\varphi : R \to R'$ be a ring homomorphism. The subring

$$\varphi^{-1}[0'] = \{r \in R \mid \varphi(r) = 0'\}$$

is the *kernel* of φ, denoted $\text{Ker}(\varphi)$.

Theorem 26.5. (Analogue of Theorem 13.15.)

Let $\varphi : R \to R'$ be a ring homomorphism and let $H = \text{Ker}(\varphi)$. Let $a \in R$. Then $\varphi^{-1}[\varphi(a)] = a + H = H + a$, where $a + H = H + a$ is the coset containing a of the commutative additive group $\langle H, +\rangle$.

Corollary 26.6. (Analogue of Theorem 13.18.)

A ring homomorphism $\varphi : R \to R'$ is a one to one map if and only if $\text{Ker}(\varphi) = \{0\}$.

Note. The following several results on factor rings are analogous to the results developed in Section 14 for factor groups.

Theorem 26.7. (Analogue of Theorem 14.1.)
Let $\varphi : R \to R'$ be a ring homomorphism with kernel H. Then the additive cosets of H form a ring R/H whose binary operations are defined by choosing representatives. That is, the sum of two cosets is defined by $(a + H) + (b + H) = (a + b) + H$ and the product of the cosets is defined by $(a + H)(b + H) = (ab) + H$. Also, the map $\mu : R/H \to \varphi[R]$ defined by $\mu(a + H) = \varphi(a)$ is an isomorphism.

Example 26.8. Consider $\varphi : \mathbb{Z} \to \mathbb{Z}_n$ where $\varphi(x) = x \pmod{n}$. This is shown to be a ring homomorphism in Example 18.11 (page 171). By Theorem 26.7, we can compute sums and products in the ring $\mathbb{Z}/n\mathbb{Z}$ ($\text{Ker}(\varphi) = n\mathbb{Z}$) using coset representatives. Also, by Theorem 26.7, we see that $\mu : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}_n$ is an isomorphism.

Note. When we defined the factor group G/H we originally based it on the kernel H of a homomorphism. Following that, we defined a factor group G/H in terms of a normal subgroup H. We now look for a condition on a subring of a ring, which will correspond to the condition on normal subgroups, so that we might generate factor rings in a setting other than that of homomorphisms.

Theorem 26.9. (Analogue of Theorem 14.4.)
Let H be a subring of the ring R. Multiplication of additive cosets of H is well defined by the equation $(a + H)(b + H) = ab + H$ if and only if $ah, hb \in H$ for all $a, b \in R$ and $h \in H$.

Note. Theorem 26.9 tells us that we can define multiplication of additive cosets if (and only if)

$$aH = \{ah \mid h \in H\} \subseteq H \text{ and } Hb = \{hb \mid h \in H\} \subseteq H$$

for all $a, b \in R$. This is the property of rings analogous to the property of normal subgroups which allowed us to produce factor groups. We'll denote subgroups with this property with an N.

Definition 26.10. An additive subgroup N of a ring R satisfying the properties $aN \subseteq N$ and $Nb \subseteq N$ for all $a, b \in R$ is an *ideal*.

Example 26.13. Let F be the ring of all functions mapping $\mathbb{R}$ into $\mathbb{R}$. Let N be the subring of F of all functions f such that $f(2) = 0$. Then N is an ideal in F. This is because for $f \in F$ and $g \in N$ we have

$$fN = \{fg \mid g \in N\} = \{f(x)g(x) \mid g(2) = 0\} \subseteq N$$

since for $f(x) \in F$, $f(2)g(2) = f(2)0 = 0$. Similarly, $Nf \subseteq N$.

Corollary 26.14. (Analogue of Corollary 14.5.)
Let N be an ideal of a ring R. Then the additive cosets of N form a ring R/N with the binary operations defined by $(a + N) + (b + N) = (a + b) + N$ and $(a + N)(b + N) = ab + N$.

Definition 26.15. The ring R/N in Corollary 26.14 is the *factor ring* (or *quotient ring*) of R by N.

Note. The following two results complete the analogies between factor groups (quotient groups) and factor rings (quotient rings).

Theorem 26.16. (Analogue of Theorem 14.9.)
Let N be an ideal of a ring R. Then $\gamma : R \to R/N$ given by $\gamma(x) = x + N$ is a ring homomorphism with kernel N.

Theorem 26.17. Fundamental Homomorphism Theorem (Analogue of Theorem 14.11.)
Let $\varphi : R \to R'$ be a ring homomorphism with kernel N. Then $\varphi[R]$ is a ring and the map $\mu : R/N \to \varphi[R]$ given by $\mu(x + N) = \varphi(x)$ is an isomorphism. If $\gamma : R \to R/N$ is the homomorphism given by $\gamma(x) = x + N$ then for each $x \in R$, we have $\varphi(x) = (\mu\gamma)(x)$.

Note. So ideals for rings are analogous to normal subgroups for groups—they allow us to define quotient rings (respectively, quotient groups). The next exercise is further evidence of this.

Exercise 26.22. Let $\varphi : R \to R'$ be a ring homomorphism and let N be an ideal of R.

(a) Then $\varphi[N]$ is an ideal of $\varphi[R]$.

(c) Let N' be an ideal either of $\varphi[R]$ or of R'. Then $\varphi^{-1}[N']$ is an ideal of R.

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