Section VI.30. Vector Spaces

Note. In this section, we repeat some of the results of Linear Algebra (MATH 2010), but instead of considering scalars which are from $\mathbb{R}$ or $\mathbb{C}$, we allow the scalars to be from any field F (which we denote in this section as $\mathbb{F}$). We use Greek letters to indicate vectors.

Definition 30.1. Let $\mathbb{F}$ be a field. A *vector space* over $\mathbb{F}$ consists of an abelian group V under addition together with an operation of scalar multiplication of each element of V be each element of $\mathbb{F}$ on the left, such that for all $a, b \in \mathbb{F}$ and $\alpha, \beta \in V$ we have:

$\mathcal{V}_1$: $a\alpha \in V$

$\mathcal{V}_2$: $a(b\alpha) = (ab)\alpha$

$\mathcal{V}_3$: $(a + b)\alpha = a\alpha + b\alpha$

$\mathcal{V}_4$: $a(\alpha + \beta) = a\alpha + a\beta$

$\mathcal{V}_5$: $1\alpha = \alpha$

The elements of V are *vectors* and the elements of $\mathbb{F}$ are *scalars*.

Note. Scalar multiplication is a function from $\mathbb{F} \times V$ into V since it maps an ordered pair (a, α) onto an element of V, denoted $a\alpha$.

Example 30.2. You are familiar with the vector space $\mathbb{R}^n$. We can develop it here by defining the abelian group $\langle \mathbb{R}^n, + \rangle = \langle \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}, + \rangle$ where addition is defined componentwise and scalar multiplication of vector $\alpha = (a_1, a_2, \ldots, a_n)$ by scalar $r \in \mathbb{R}$ is defined as

$$r\alpha = r(a_1, a_2, \ldots, a_n) = (ra_1, ra_2, \ldots, ra_n).$$

We can similarly develop $\mathbb{C}^n$.

Example 30.3. For field $\mathbb{F}$, $\mathbb{F}[x]$ can be viewed as a vector space. The vectors are then

$$\sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

where all but finitely many of the a_i are 0. Vector addition and scalar multiplication are dealt with as with polynomials (see Section 22). Notice that we loose some structure by considering $\mathbb{F}[x]$ as a vector space instead of a ring of polynomials. We cannot multiply vectors together in a vector space, but we can multiply polynomials together in a ring of polynomials.

Example 30.4. Let E be an extension field of a field F. In the next section, we will make extensive use of the fact that E can be interpreted as a vector space over F. The addition of vectors (elements of E) is the usual addition in E and scalar multiplication α ($a \in F$, $\alpha \in E$) is the usual multiplication in E.

Theorem 30.5. If V is a vector space over field F, then $0\alpha = 0$, $a0 = 0$, and $(-a)\alpha = a(-\alpha) = -(a\alpha)$ for all $a \in F$ and $\alpha \in V$.

Definition 30.6. Let V be a vector space over field F. The vectors in subset $S = \{\alpha_i \mid i \in I\}$ of V span (or *generate*) V if for every $\beta \in V$ we have

$$\beta = a_1\alpha_{i_1} + a_2\alpha_{i_2} + \cdots + a_n\alpha_{i_n}$$

for some $a_j \in F$ and $\alpha_{i_j} \in S$, $j = 1, 2, \ldots, n$. A vector $\sum_{j=1}^n a_j\alpha_{i_j}$ is a *linear combination* of the α_{i_j}.

Example 30.8. Let E be an extension field of field F. Let $\alpha \in E$ be algebraic over F of degree n. Then $F(\alpha)$ is a vector space over F and by Theorem 29.18 this vector space is spanned by the vectors in $\{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}\}$. This will allow us to talk about the dimension of E over F.

Definition 30.9. A vector space V over a field F is *finite dimensional* if there is a finite subset of V whose vectors span V.

Example 30.10. The vector space $F[x]$ over field F is *not* finite dimensional since polynomials of arbitrarily large degree could not be linear combinations of elements of any finite set of polynomials.

Definition 30.12. The vectors in a subset $S = \{\alpha_i \mid i \in I\}$ of a vector space V over a field F are *linearly independent* over F if, for any distinct vectors $\alpha_{i_j} \in S$, any coefficients $a_j \in F$ and any $n \in \mathbb{N}$ we have $\sum_{j=1}^n a_j\alpha_{i_j} = 0$ in V only if $a_j = 0$ for $j = 1, 2, \ldots, n$. If the vectors in S are not linearly independent over F then they are *linearly dependent* over F.

Example 30.14. Let E be an extension field of field F. Let $\alpha \in E$ be algebraic over F. If $\deg(\alpha, F) = n$ then by Theorem 29.18, every element of $F(\alpha)$ can be uniquely written in the form

$$b_0 + b_1\alpha + b_2\alpha^2 + \cdots + b_{n-1}\alpha^{n-1}$$

for $b_i \in F$. Since $0 \in F(\alpha)$, we have that $0 = b_1 + b_1\alpha + b_2\alpha^2 + \cdots + b_{n-1}\alpha^{n-1}$ implies (by the uniqueness) that $b_0 = b_1 = b_2 = \cdots = b_{n-1} = 0$. So the vectors $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$ are linearly independent over F. Since they also span $F(\alpha)$ (Theorem 29.18 again) then they form a basis for $F(\alpha)$ over F (and "basis" is defined next).

Definition 30.15. If V is a vector space over a field F, the vectors in a subset $B = \{\beta_i \mid i \in I\}$ of V form a *basis* for V over F if they span V and are linearly independent.

Note. In the next few results, we establish that a finite dimensional vector space has a basis. This is also true for infinite dimensional vector spaces, but requires Zorn's Lemma. For details, see my notes on Introduction to Functional Analysis (MATH 5740) from *Real Analysis with an Introduction to Wavelets and Applications*, by D. Hong, J. Wang, and R. Gardner (Elsevier Press, 2005): `http://faculty.etsu.edu/gardnerr/Func/notes/HWG-5-1.pdf` (see page 7, Theorem 5.1.4).

Lemma 30.16. Let V be a vector space over a field F, and let $\alpha \in V$. If α is a linear combination of vectors β_i in V for $i = 1, 2, \ldots, m$ and each β_i is a linear combination of vectors γ_j for $j = 1, 2, \ldots, n$, then α is a linear combination of the γ_j.

Theorem 30.17. In a finite dimensional vector space, every finite set of vectors spanning the space contains a subset that is a basis.

Corollary 30.18. A finite dimensional vector space has a finite basis.

Theorem 30.19. Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ be a finite set of linearly independent vectors in a finite dimensional vector space V over a field F. Then S can be enlarged to a basis for V over F. Furthermore, if $B = \{\beta_1, \beta_2, \ldots, \beta_n\}$ is any basis for V over F, then $r \leq n$.

Note. The following result allows us to define the dimension of a vector space. We can then reword Theorem 30.19 as "A linearly independent set of r vectors in a vector space of dimension n satisfies $r \leq n$." This result can also be arrived at by considering systems of equations, instead of the "casting out" technique of Theorem 30.17. For details, see notes used in Introduction to Functional Analysis (MATH 5740) from *Real Analysis with an Introduction to Wavelets and Applications*, by D. Hong, J. Wang, and R. Gardner (Elsevier Press, 2005), section 5.1 "Groups, Fields, and Vector Spaces" (`http://faculty.etsu.edu/gardnerr/Func/notes/HWG-5-1.pdf`).

Corollary 30.20. Any two bases of a finite-dimensional vector space V over F have the same number of elements.

Proof. Let $B = \{\beta_1, \beta_2, \ldots, \beta_n\}$ and $B' = \{\beta'_1, \beta'_2, \ldots, \beta'_m\}$ be two bases. By Theorem 30.19, B is an independent set and B' a basis, so $n \leq m$. Interchanging the roles of B and B' gives $m \leq n$. Therefore $m = n$. ∎

Note. Corollary 30.20 also holds for vector spaces of infinite dimensions. See Section 5.1 Exercise 3 of *Real Analysis with an Introduction to Wavelets and Applications*, by D. Hong, J. Wang, and R. Gardner (Elsevier Press, 2005).

Definition 30.21. If V is a finite-dimensional vector space over a field F, the number of elements in a basis is the *dimension* of V over F.

Note. You see in Linear Algebra (MATH 2010) that for vector space V over scalar field $\mathbb{F}$, the vector space is dimension n if and only if V is isomorphic to $\mathbb{F}^n$. This is sometimes called the *Fundamental Theorem of Finite Dimensional Vector Spaces*. See Theorem 5.1.2 of the functional analysis notes mentioned above (`http://faculty.etsu.edu/gardnerr/Func/notes/HWG-5-1.pdf`).

Example 30.22. Let E be an extension field of field F and let $\alpha \in E$. If α is algebraic over F and $\deg(\alpha, F) = n$, then by Example 30.14, a basis for $F(\alpha)$ is $\{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}\}$, so the dimension of $F(\alpha)$ treated as a vector space over F is n.

Note. We now collect the claims of several of the above examples together and give a proof of one new claim.

Theorem 30.23. Let E be an extension field of field F and let $\alpha \in E$ be algebraic over F. If $\deg(\alpha, F) = n$, then $F(\alpha)$ is an n-dimensional vector space over F with basis $\{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}\}$. Furthermore, every element β of $F(\alpha)$ is algebraic over F, and $\deg(\beta, F) \leq \deg(\alpha, F)$.

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