Part VII. Advanced Group Theory

Section VII.34. Isomorphism Theorems

Note. In this section we explore various relationships between groups and factor groups (i.e., quotient groups). This is accomplished in three "isomorphism theorems."

Note. The following result involves the kernel of a homomorphism. This is how we first generated factor groups (groups of cosets) in Section 14.

Theorem 34.2. First Isomorphism Theorem.
Let $\phi : G \to G'$ be a homomorphism between multiplicative groups G and G'. Let $K = \text{Ker}(\phi)$ and let $\gamma_K : G \to G/K$ be the canonical homomorphism $\gamma_K(g) = gK$ for $g \in G$. There is a unique isomorphism $\mu : G/K \to \phi[G]$ such that $\phi(g) = \mu(\gamma_K(g)) = \mu(gK)$ for all $g \in G$.

Note. We represented the First Isomorphism Theorem diagramatically in Section 13 (see page 6 of the class notes).

Note. We need some preliminary results before proving the other two isomorphism theorems.

Lemma 34.3. Let N be a normal subgroup of a group G and let $\gamma : G \to G/N$ be the canonical homomorphism. Then the map ϕ from the set of normal subgroups of G containing N to the set of normal subgroups of G/N given by $\phi(L) = \gamma[L]$ is one to one and onto.

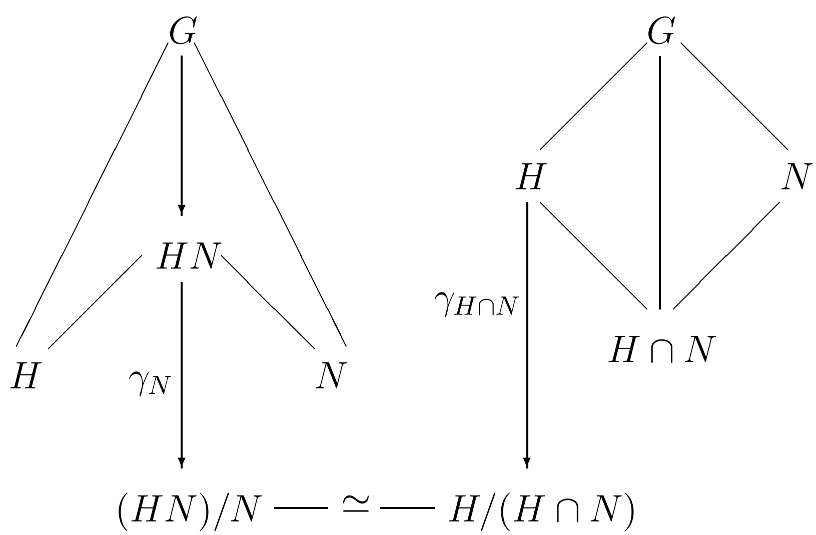
Definition. If H and N are subgroups of a group G, then let $HN = \{hn \mid h \in H, n \in N\}$. The *join* of H and N is the intersection of all subgroups of G that contain set HN. This is denoted $H \vee N$.

Note. By Theorem 7.4, $H \vee N$ is a subgroup of G. By definition, it is "the smallest" subgroup containing both H and N (the term "smallest" is used in a set inclusion sense).

Lemma 34.4. If N is a normal subgroup of G, and if H is any subgroup of G, then $H \vee N = HN = NH$. Furthermore, if H is also normal in G, then HN is normal in G.

Theorem 34.5. Second Isomorphism Theorem.
Let H be a subgroup of G and let N be a normal subgroup of G. Then $(HN)/N \simeq H/(H \cap N)$.

Note. As a diagram, we might represent the Second Isomorphism Theorem as follows.



Note. As seen in the proof of the Second Isomorphism Theorem, the isomorphism between $(HN)/N$ and $H/(H \cap N)$ is given by

$$\phi((hn)N) = \mu_1^{-1}(\mu_2((hn)N)) = \mu_1^{-1}(h) = h(H \cap N),$$

or in additive notation,

$$\phi((h+n)+N) = \mu_1^{-1}(\mu_2((h+n)+N)) = \mu_1^{-1}(h) = h + H \cap N$$

where $\mu_1 : H/(H \cap N) \to \gamma[H]$ and $\mu_2 : (HN)/N \to \gamma[H]$ are isomorphisms.

Exercise 34.4. For $G = \mathbb{Z}_{36}$, let $H = \langle 6 \rangle$ and $N = \langle 9 \rangle$.

(a) List the elements in HN (which should be written $H + N$ since these are additive groups) and $H \cap N$.

Solution. We have $H = \langle 6 \rangle = \{0, 6, 12, 18, 24, 30\}$ and $N = \langle 9 \rangle = \{0, 9, 18, 27\}$. So $H + N = \{0, 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33\}$ and $H \cap N = \{0, 18\}$.

(b) List the cosets of $(HN)/N = (H + N)/N$.

Solution. The cosets are $N + 0 = N + 9 = N + 18 = N + 27 = \{0, 9, 18, 27\}$, $N + 3 = N + 12 = N + 21 = N + 30 = \{3, 12, 21, 30\}$, and $N + 6 = N + 15 = N + 24 = N + 33 = \{6, 15, 24, 33\}$.

(c) List the cosets of $H/(H \cap N)$.

Solution. The cosets are $H \cap N + 0 = H \cap N + 18 = \{0, 18\}$, $H \cap N + 6 = H \cap N + 24 = \{6, 24\}$, and $H \cap N + 12 = H \cap N + 30 = \{12, 30\}$.

(d) Give the correspondence between $(HN)/N = (H + N)/N$ and $H/(H \cap N)$ described in the proof of Theorem 34.5.

Solution. In the notation of Theorem 34.5, we have $\phi : (H + N)/N \to H/(H \cap N)$ where $\phi((h + n) + N) = h + (H \cap N)$, so $\phi((0 + 0) + N) = \phi((0+9)+N) = \phi((0+18)+N) = \phi((0+27)+N) = 0+H\cap N$, $\phi((6+0)+N) = \phi((6 + 9) + N) = \phi((6 + 18) + N) = \phi((6 + 27) + N) = 6 + H \cap N$, and $\phi((12 + 0) + N) = \phi((12 + 9) + N) = \phi((12 + 18) + N) = \phi((12 + 27) + N) = 12 + H \cap N$. So ϕ gives $\{0, 9, 18, 27\} \to \{0, 18\}$, $\{6, 15, 24, 33\} \to \{6, 24\}$, and $\{3, 12, 21, 30\} \to \{12, 30\}$.

Theorem 34.7. Third Isomorphism Theorem.

Let H and K be normal subgroups of a group G with K a subgroup of H. Then $G/H \simeq (G/K)/(H/K)$.

Note. The Third Isomorphism Theorem (Theorem 34.7) can be viewed as performing the "collapse" of G down to G/H in two steps. First, collapse G to G/K (since $K \leq H$, then we expect G/K to be "bigger" than G/H), and second collapse G/K down to $(G/K)/(H/K)$. In terms of the canonical mappings, we have $\gamma_H = \gamma_{H/K}\gamma_K$, as in the diagram below.

$$\begin{tikzcd}
G \arrow[r, dash, "="] \arrow[d, "\gamma_K"'] & G \arrow[dd, "\gamma_H"] \\
G/K \arrow[d, "\gamma_{H/K}"'] & \\
(G/K)/(H/K) \arrow[r, dash, "\simeq"] & G/H
\end{tikzcd}$$

Exercise 34.5. Let $G = \mathbb{Z}_{24}$, $H = \langle 4 \rangle$, and $K = \langle 8 \rangle$.

(a) List the cosets in G/H.

Solution. We have $H = \langle 4 \rangle = \{0, 4, 8, 12, 16, 20\}$ and $K = \langle 8 \rangle = \{0, 8, 16\}$. The cosets are $H + 0 = H + 4 = H + 8 = H + 12 = H + 16 = H + 20 = \{0, 4, 8, 12, 16, 20\}$, $H + 1 = H + 5 = H + 9 = H + 13 = H + 17 = H + 21 = \{1, 5, 9, 13, 17, 21\}$, $H + 2 = H + 6 = H + 10 = H + 14 = H + 18 = H + 22 = \{2, 6, 10, 14, 18, 22\}$, and $H + 3 = H + 7 = H + 11 = H + 15 = H + 19 = H + 23 = \{3, 7, 11, 15, 19, 23\}$.

(b) List the cosets in G/K.

Solution. The cosets are $K + 0 = K + 8 = K + 16 = \{0, 8, 16\}$, $K + 1 = K + 9 = K + 17 = \{1, 9, 17\}$, $K + 2 = K + 10 = K + 18 = \{2, 10, 18\}$, $K+3 = K+11 = K+19 = \{3, 11, 19\}$, $K+4 = K+12 = K+20 = \{4, 12, 20\}$, $K+5 = K+13 = K+21 = \{5, 13, 21\}$, $K+6 = K+14 = K+22 = \{6, 14, 22\}$, and $K + 7 = K + 15 = K + 23 = \{7, 15, 23\}$.

(c) List the cosets in H/K.

Solution. The cosets are $K + 0 = K + 8 = K + 16 = \{0, 8, 16\}$, and $K + 4 = K + 12 = K + 20 = \{4, 12, 20\}$.

(d) List the cosets in $(G/K)/(H/K)$.

Solution. The cosets are of the form (element of G/K) + (H/K), since H/K consists of two cosets, we have that the cosets are $(K + 0) + (H/K) = (K + 4) + (H/K) = \{0, 8, 16\} + \{\{0, 8, 16\}, \{4, 12, 20\}\} = \{\{0, 8, 16\}, \{4, 12, 20\}\}$, $(K + 1) + (H/K) = (K + 5) + (H/K) = \{1, 9, 17\} + \{\{0, 8, 16\}, \{4, 12, 20\}\} = \{\{1, 9, 17\}, \{5, 13, 21\}\}$, $(K + 2) + (H/K) = (K + 6) + (H/K) = \{2, 10, 18\} +$

$\{\{0,8,16\},\{4,12,20\}\} = \{\{2,10,18\},\{6,14,22\}\}$, and $(K+3)+(H/K) = (K+7)+(H/K) = \{3,11,19\}+\{\{0,8,16\},\{4,12,20\}\} = \{\{3,11,19\},\{7,15,23\}\}$.

(e) Give the correspondence ("natural isomorphism") between G/H and $(G/K)/(H/K)$.

Solution. The natural isomorphism maps $H+i$ to $(K+i)+H/K$ for $i = 0,1,2,3$. So $\{0,4,8,12,16,20\} \to \{\{0,8,16\},\{4,12,20\}\}$, $\{1,5,9,13,17,21\} \to \{\{1,9,17\},\{5,13,21\}\}$, $\{2,6,10,14,18,22\} \to \{\{2,10,18\},\{6,14,22\}\}$, and $\{3,7,11,15,19,23\} \to \{\{3,11,19\},\{7,15,23\}\}$ (where $i = 0,1,2,3$, respectively).

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