Section VII.35. Series of Groups

Note. In this section we introduce a way to consider a type of factorization of a group into simple factor groups. This idea is similar to factoring a natural number into prime factors. The big result of this section is the Jordan-Hölder Theorem which gives an indication of *why* it is important to classify finite simple groups (first encountered in Section III.15).

Definition 35.1. A *subnormal* (or *subinvariant*) *series* of a group G is a finite sequence $H_0, H_1, \ldots, H_n$ of subgroups of G such that $H_i < H_{i+1}$ and H_i is a normal subgroup of H_{i+1} (i.e., $H_i \triangleleft H_{i+1}$) with $H_0 = \{e\}$ and $H_n = G$. A *normal* (or *invariant*) *series* of group G is a finite sequence $H_0, H_1, \ldots, H_n$ of normal subgroups of G (i.e., $H_i \triangleleft G$) such that $H_i < H_{i+1}$, $H_0 = \{e\}$, and $H_0 = G$.

Note. If H_i is normal in G, then $gH_i = H_ig$ for all $g \in G$ and so H_i is normal in $H_{i+1} \leq G$. So every normal series of G is also a subnormal series of G.

Note. If G is an abelian group, then all subgroups of G are normal, so in this case there is no distinction between a subnormal series and a normal series.

Example. A normal series of $\mathbb{Z}$ is:

$$\{0\} < 16\mathbb{Z} < 8\mathbb{Z} < 4\mathbb{Z} < \mathbb{Z}.$$

Example 35.3. The dihedral group on 4 elements D_4 has the subnormal series:

$$\{\rho_0\} < \{\rho_0, \mu_1\} < \{\rho_0, \rho_2, \mu_1, \mu_2\} < D_4.$$

This is not a normal series since $\{\rho_0, \mu_1\}$ is not a normal subgroup of D_4. From Table 8.12 (page 80) we see that

$$\delta_1\{\rho_0, \mu_1\} = \{\delta_1, \rho_1\} \neq \{\delta_1, \rho_3\} = \{\rho_0, \mu_1\}\delta_1.$$

Definition 35.4. A subnormal (normal) series $\{K_j\}$ is a *refinement* of a subnormal (normal) series $\{H_i\}$ of a group G if $\{H_i\} \subseteq \{K_j\}$, that is, if each H_i is one of the K_j.

Example. A refinement of the normal series of $\mathbb{Z}$ given above is

$$\{0\} < 64\mathbb{Z} < 32\mathbb{Z} < 16\mathbb{Z} < 8\mathbb{Z} < 4\mathbb{Z} < 2\mathbb{Z} < \mathbb{Z}.$$

Definition 35.6. Two subnormal (normal) series $\{H_i\}$ and $\{K_j\}$ of the same group G are *isomorphic* if there is a one-to-one correspondence between the collection of factor groups $\{H_{i+1}/H_i\}$ and $\{K_{j+1}/K_j\}$ such that corresponding factor groups are isomorphic.

Note. The one-to-one correspondence implies that the sets $\{H_{i+1}/H_i\}$ and $\{K_{j+1}/K_j\}$ are of the same cardinality. The "corresponding factor groups are isomorphic" does not imply that the factor groups are isomorphic *in order*, but that the correspondence is given by the one-to-one correspondence.

Example 35.7. The two series $\{0\} < \langle 5\rangle < \mathbb{Z}_{15}$ and $\{0\} < \langle 3\rangle < \mathbb{Z}_{15}$ are isomorphic normal series since the set of factor groups for $\{0\} < \langle 5\rangle < \mathbb{Z}_{15}$ is $\{\mathbb{Z}_{15}/\langle 5\rangle \simeq \mathbb{Z}_5, \langle 5\rangle/\langle 0\rangle \simeq \mathbb{Z}_3\}$ and the set of factor groups for $\{0\} < \langle 3\rangle < \mathbb{Z}_{15}$ is $\{\mathbb{Z}_{15}/\langle 3\rangle \simeq \mathbb{Z}_3, \langle 3\rangle/\langle 0\rangle \simeq \mathbb{Z}_5\}$.

Note. The Schreier Theorem states that any two normal (or subnormal) series of a group G have refinements which are isomorphic. First, we illustrate this with an example and then we prove a lemma we will use in the proof of the Schreier Theorem.

Example 35.8. Consider the two normal series of $\mathbb{Z}$: (1) $\{0\} < 8\mathbb{Z} < 4\mathbb{Z} < \mathbb{Z}$ and (2) $\{0\} < 9\mathbb{Z} < \mathbb{Z}$. consider the refinement of (1) $\{0\} < 72\mathbb{Z} < 8\mathbb{Z} < 4\mathbb{Z} < \mathbb{Z}$ and the refinement of (2) $\{0\} < 72\mathbb{Z} < 18\mathbb{Z} < 9\mathbb{Z} < \mathbb{Z}$. The four factor groups for both refinements are

$$72\mathbb{Z}/\{0\} \cong 72\mathbb{Z},\ 8\mathbb{Z}/72\mathbb{Z} \cong \mathbb{Z}/9\mathbb{Z} \cong \mathbb{Z}_9,$$

$$4\mathbb{Z}/8\mathbb{Z} \cong 9\mathbb{Z}/18\mathbb{Z} \cong \mathbb{Z}_2,\ \mathbb{Z}/4\mathbb{Z} \cong 18\mathbb{Z}/72\mathbb{Z} \cong \mathbb{Z}_4.$$

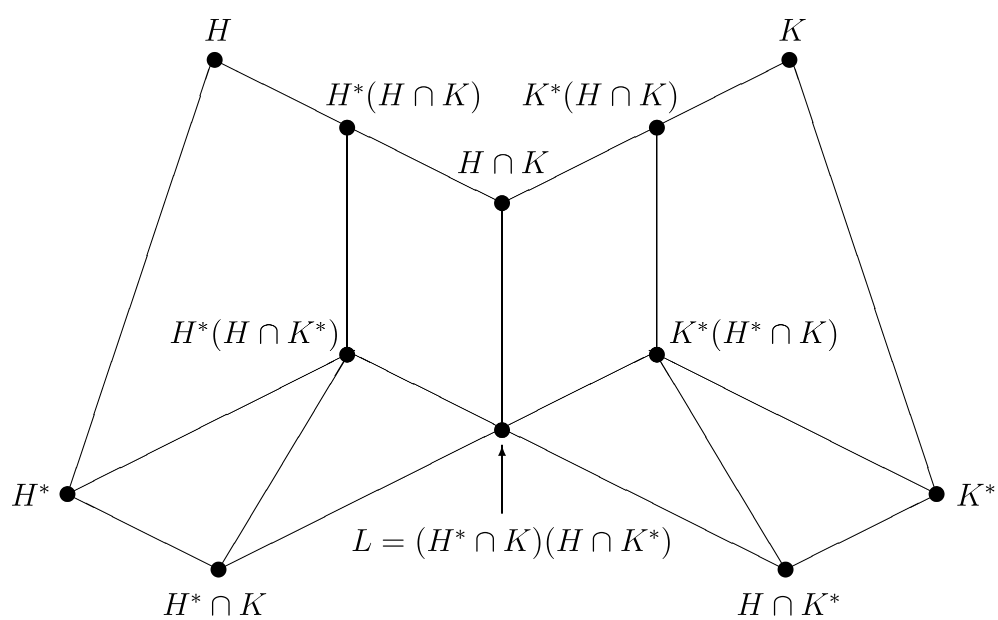
Notice the factor groups are the same, although they appear in different orders. So there is a one to one correspondence between the factor groups $\{H_{i+1}/H_i\}$ and $\{K_{j+1}/K_j\}$. That is, the refinements are isomorphic.

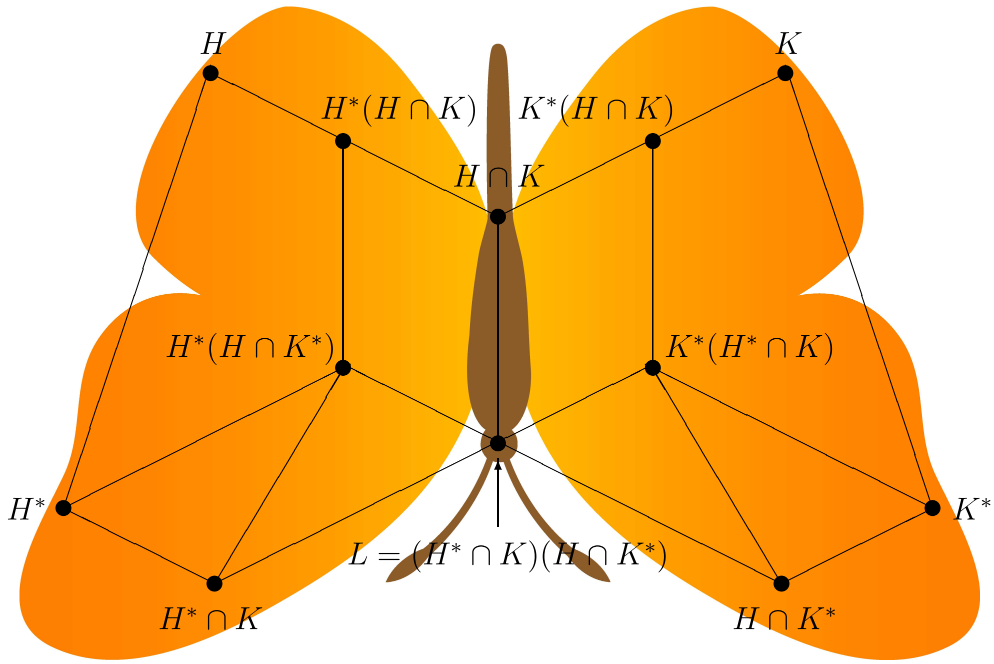
Lemma 35.10. Zassenhaus Lemma/Butterfly Lemma.

Let H and K be subgroups of a group G and let H^* and K^* be normal subgroups of H and K respectively. Then

1. $H^*(H \cap K^*)$ is a normal subgroup of $H^*(H \cap K)$,
2. $K^*(H^* \cap K)$ is a normal subgroup of $K^*(H \cap K)$, and
3. $$H^*(H \cap K)/H^*(H \cap K^*) \cong K^*(H \cap K)/K^*(H^* \cap K)$$ $$\cong (H \cap K)/[(H^* \cap K)(H \cap K^*)].$$

Note. The following diagrams indicate which groups are subgroups of which, and therefore why the Zassenhaus Lemma in sometimes called the Butterfly Lemma. On the surface, there is no motivation for the Zassenhaus Lemma. However, we will see why we need it in the Schreier Theorem. Now for the proof of the Zassenhaus Lemma.





Note. The proof of the Schreier Theorem is *constructive*. That is, the existence of an object is claimed and then established by actually building the object.

Theorem 35.11. Schreier Theorem.

Two subnormal (normal) series of a group G have isomorphic refinements.

Definition 35.12. A subnormal series $\{H_i\}$ of a group G is a *composition series* if all the factor groups H_{i+1}/H_i are simple. A normal series $\{H_i\}$ of G is a *principal* (or *chief*) *series* if all the factor groups H_{i+1}/H_i are simple.

Note. If G is an abelian group, then all subgroups are normal and so there is no difference in a normal series and a subnormal series. So, for an abelian group, composition series and principal series are the same. For any group G, a normal series is also a subnormal series, so every principal series is a composition series.

Exercise 35.13. Consider the group $\mathbb{Z}$. We claim there is no composition series (and hence no principal series, since $\mathbb{Z}$ is abelian). If $\{0\} = H_0 < H_1 < \ldots < H_{n-1} < H_n = \mathbb{Z}$ is a subnormal series, then H_1 is a proper, nontrivial subgroup of $\mathbb{Z}$. So $H_1 = r\mathbb{Z}$ for some $r \in \mathbb{N}$, by Corollary 6.7. But then $H_1/H_0 = r\mathbb{Z}/\{e\} \cong r\mathbb{Z}$. However, for the above chain to be a composition series, we must have H_1/H_0 simple (that is, it has no proper, nontrivial normal subgroup). But $(2r)\mathbb{Z}$ is a proper, nontrivial subgroup of $r\mathbb{Z}$, which is normal since $r\mathbb{Z}$ is abelian. Therefore there is no composition series for $\mathbb{Z}$.

Example 35.14. The series $\{e\} < A_n < S_n$ for $n \geq 5$ is a composition series of S_n because $A_n/\{e\} \cong A_n$ is simple for $n \geq 5$ by Exercise 15.39, and $S_n/A_n \cong \mathbb{Z}_2$, which is simple. Notice $\{e\}$ is a normal subgroup of A_n and of S_n. Also, A_n is a normal subgroup of S_n by Exercise 14.24. So the series is both a composition series and a principal series.

Note. Recall that (by definition) a maximal normal subgroup of a group G is a normal subgroup $M \neq G$ such that there is no proper normal subgroup N of G properly containing M. By Theorem 15.18, M is a maximal normal subgroup of G if and only if G/M is simple. So for a composition series of group G, each H_i in the chain must be a maximal subgroup of H_{i+1}. Hence, to find a composition series of group G, we need a maximal normal subgroup H_{n-2} of H_{n-1}, and so on until the process terminates in a *finite number of steps* at $\{e\}$. Since the series consists of maximal subgroups, a composition series cannot be refined. This does not imply it is unique since at each step, the maximal normal subgroup may not be unique. To find a principal series, we similarly look for maximal normal subgroup H_i in H_{i+1}, but also need H_i to be normal in G.

Example. Two composition series (and principal series) for $\mathbb{Z}_6$ are $\{0\} < \{0, 2, 4\} < \mathbb{Z}_6$ and $\{0\} < \{0, 3\} < \mathbb{Z}_6$. However these two series are isomorphic since both have associated factor groups (isomorphic to) $\mathbb{Z}_2$ and $\mathbb{Z}_3$. This is no coincidence, as shown in the following (the "real meat" of this section, according to Fraleigh).

Theorem 35.15. Jordan-Hölder Theorem.

Any two composition series (or principal series) of a group G are isomorphic.

Proof. Let $\{H_i\}$ and $\{K_i\}$ be two composition series (or principal series) of G. By Theorem 35.11, they have isomorphic refinements. But since all associated factor groups are simple, by Theorem 15.18 (as commented above), there is no refinement of the series. So $\{H_i\}$ and $\{K_i\}$ must already be isomorphic. ∎

Note. In the supplement to the notes from Introduction to Modern Algebra 1 on "Simple Groups" also mentioned the Jordan-Hölder Theorem. The statement then was:

Jordan-Hölder Theorem. Every finite group G has a *composition series*

$$\{e\} = G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \cdots \triangleleft G_{n-1} \triangleleft G_n = G$$

where each group is normal in the next, and the series cannot be refined any further: in other words, each G_i/G_{i-1} is *simple.*

This statement is from Robert S. Wilson's *The Finite Simple Groups*, *Graduate Texts in Mathematics 251* (New York: Springer Verlag, 2009). This version shows the importance of simple groups and the sense in which they are like prime numbers: Finite group G is "composed" of the groups in the compositions series and the series cannot be further refined (just as a prime number cannot be further factored). I say "composed," but this term is used in the sense explained in the theorem. It is tempting to extend the idea further than is allowed (see the next comment)!

Note. Fraleigh describes a composition series "as a type of factorization" of a group into simple factor groups. However, the term "factorization" should not be taken in a sense of direct "products." For example, a composition series for $\mathbb{Z}_4$ is $\{0\} < \{0,2\} < \mathbb{Z}_4$ with associated factor groups (isomorphic to) $\{0,2\}/\{0\} \cong \mathbb{Z}_2$ and $\mathbb{Z}_4/\{0,2\} \cong \mathbb{Z}_2$. However,

$$(\mathbb{Z}_4/\{0,2\}) \times (\{0,2\}/\{0\}) \ncong \mathbb{Z}_4/\{0\} \cong \mathbb{Z}_4$$

since the left hand side if $\mathbb{Z}_2 \times \mathbb{Z}_2$ which is isomorphic to the Klein 4-group. It is tempting to think that the $\{0,2\}$'s "cancel," but this is not the case. The simple factor groups are not "factors" in this sense of being parts of a direct product.

Theorem 35.16. If G has a composition series (or principal series) and if N is a proper normal subgroup of G, then there exists a composition series (principal series) containing N.

Note. The following concept will play a role in reaching our "final goal" which concerns solving polynomial equations with radicals in Section X.56.

Definition 35.18. A group G is *solvable* if it has a composition series $\{H_i\}$ such that all factor groups H_{i+1}/H_i are abelian.

Note. A solvable group has *a* composition series where all factor groups are abelian. By the Jordan-Hölder Theorem, all composition series of a solvable group have abelian factor groups.

Note. Our "final goal" was first proved by Niels Henrik Abel (1802–1829) in 1823. The fact that H_{i+1}/H_i are commutative is key to the proof. This is the reason that Abel's name is associated with commutative ("abelian") groups.

Example 35.19. The group S_3 is solvable because the composition series $\{e\} < A_3 < S_3$ has factor groups $A_3/\{e\} \cong A_3 \cong \mathbb{Z}_3$ and $S_3/A_3 \cong \mathbb{Z}_2$, both of which are abelian. The group S_5 is not solvable since the series $\{e\} < A_5 < S_5$ is a composition series (because $A_5/\{e\} \cong A_5$ is simple by Theorem 15.15, and $S_5/A_5 \cong \mathbb{Z}_2$ is simple [it has no proper nontrivial normal subgroups]) but $A_5/\{e\} \cong A_5$ is not abelian. In addition, A_5 (or order $5!/2 = 60$) is also not solvable and is the smallest nonsolvable group. We will use the fact that A_5 is not solvable later to show the "final goal" of the insolvability of the quintic.

Revised: 3/20/2014