Section VII.39. Free Groups

Note. In this section, we define "free group," in general (not just for abelian groups) and define the rank of such groups. The importance of this class of groups is illustrated in Theorem 39.13 in which it is shown that every group if a homomorphic image of some free group.

Note. Surprisingly, Fraleigh recommends Crowell and Fox's *Introduction to Knot Theory* (1963) as supplemental reading for this section and the next section (on group presentations). "Knot Theory" is a branch of topology. Connections between algebra and topology are given in Fraleigh's Sections 41–44.

Note. As Fraleigh mentions on page 341, we will gloss over some of the more tedious details (of rather unsurprisingly results) in this section. A more detailed treatment of these ideas can be found in Hungerford's *Algebra* (Section I.9) and David Dummit and Richard Foote's *Abstract Algebra*, 3rd Edition (John Wiley and Sons, 2004), Section 6.3.

Definition. Let A be a set, say $a = \{a_i \mid i \in I\}$ for some indexing set I. Then set A is an *alphabet* and elements a_i are *letters* in the alphabet. A symbol of the form a_i^n with $n \in \mathbb{Z}$ is a *syllable*. A finite "string" of syllables is a *word*. The *empty word* is denoted 1.

Example. With $A = \{a, b, c\}$, we have $w = a^5a^{-4}b^1b^2a^{-2}a^1c^2c^{-2}$ is a word.

Definition. An *elementary contraction* of a word consists of either replacing the syllables $a_i^m a_i^n$ with a_i^{m+n} or by replacing a_i^0 with 1 (or simply by dropping it from the word). A *reduced word* is a word which does not admit any more elementary contractions.

Example. The word in the previous example can be "elementary contracted" to the reduced word $a^1b^3a^{-1}$.

Note. We denote the set of all reduced words from alphabet A as $F[A]$. We define a binary operation on $F[A]$ by taking two words $w_1, w_2 \in F[A]$ and defining $w_1 \cdot w_2$ as the reduced word which results by applying elementary contractions to the word w_1w_2 which consists of the syllables of w_1 written in order before the syllables of w_2 (the "juxtaposition" of w_1 and w_2).

Example 39.3. If $w_1 = a_2^3a_1^{-5}a_3^2$ and $w_2 = a_3^{-2}a_1^2a_3a_2^{-2}$, then

$$w_1 \cdot w_2 = (a_2^3a_1^{-5}a_3^2)(a_3^{-2}a_1^2a_3a_2^{-2}) = a_2^3a_3a_2^{-2}.$$

Notice that we may eliminate the superscript when it is 1.

Note. As Fraleigh says on page 342, "it would seem obvious" that the binary operation yields a group $\langle F[A], \cdot\rangle$.

Definition 39.4. The group $F[A]$ above is the *free group generated by* A.

Note. Recall (Definition 7.5, page 69) that for group G with $\{a_i \mid i \in I\} \subseteq G$, the smallest subgroup of G containing $\{a_i \mid i \in I\}$ is the *subgroup generated by* $\{a_i \mid i \in I\}$. If this group is all of G then set $\{a_i \mid i \in I\}$ generates group G and the elements a_i are generators of group G.

Definition 39.5. If G is a group with a set $A\{a_i\}$ of generators and if G is isomorphic to $F[A]$ under the map $\phi : G \to F[A]$ such that $\phi(a_i) = a_i$, then group G is *free on* A and the a_i are *free generators of* G. A group is *free* if it is free on some nonempty set A.

Example 39.6. $\mathbb{Z}$ is a free group with one generator (say a). Notice that every free group is infinite since the syllables a^n are all different for $n \in \mathbb{Z}$ (and so the corresponding words are reduced and distinct).

Note. We now state three results without proof.

Theorem 39.7. If group G is free on A and also on B, then the sets A and B have the same cardinality.

Definition 39.8. If G is free on A, the cardinality of A is the *rank of the free group* G.

Theorem 39.9. Two free groups are isomorphic if and only if they have the same rank.

Theorem 39.10. A nontrivial proper subgroup of a free group is free.

Note. Theorems 39.7, 39.9, and 39.10 also hold for free abelian roups (see Fraleih, page 344).

Example 39.11. Let $F[\{x, y\}]$ be the free group on set $A = \{x, y\}$. Let $y_k = x^k y x^{-k}$ for $k \in \mathbb{Z}$, $k \geq 0$. Define $B = \{y_k \mid y \in \mathbb{Z}, y \geq 0\}$. Then $F[B]$ is a subgroup of $F[A]$. However, the rank of $F[B]$ is infinite even though the rank of $F[A]$ is 2. So the rank of a free group and its subgroup may not behave like the rank of a free abelian group and the rank of one of its subgroup (compare to Theorem 38.11).

Note. We now give two theorems with proof. The second one shows some of the importance of free groups.

Theorem 39.12. Let G be a group generated by $A = \{a_i \mid i \in I\}$ and let G' be any group. If a_i' for $i \in I$ are any elements in G', not necessarily distinct, then there is at most one homomorphism $\phi :\to G'$ such that $\phi(a_i) = a_i'$. If G is free on A, then there exists exactly one such homomorphism.

Theorem 39.13. Every group G' is a homomorphic image of a free group G.

Note. Fraleigh misses an "every group" opportunity here. The following result follows from equipment we already have.

Theorem. Gallian's "Universal Quotient Group Property."
Every group is isomorphic to a quotient group of a free group.

Note. As observed in Example 39.6, a free group with one generator is isomorphic to $\mathbb{Z}$ (ans do is abelian). If a free group has two (or more) generators, say $A = \{a, b, \ldots\}$, then the free group $F[A]$ is not abelian since reduced word ab and reduced word ba are different (reduced words are the same only when one can be converted into the other using the two elementary contractions on page 341). So there is quite a bit of difference in a *free abelian group* (which has a *basis* and "rank" as described in Section 38) and a *free group* (which has a *generating set* and also "rank"). However, Theorem 39.13 gives a way to relate free abelian groups to free groups.

Note. Let $F[A]$ be the free group generated by set A. Let C be the commutator subgroup of $F[A]$ (that is, C is the smallest normal subgroup of $F[A]$ containing all commutators $aba^{-1}b^{-1} \in F[A]$; see Theorem 15.20). Notice that if $|A| = 1$, then $F[A] \cong \mathbb{Z}$ is abelian and $C = \{e\} = \{0\}$. In any case, by Theorem 15.20, $F[A]/C$ is abelian (and, of course, if $F[A]$ is abelian then $F[A]/C \cong F[A]$). Now for any element of $F[A]/C$, say fC where $f \in F[A]$, we have $f = \prod_j (a_{i_j})^{n_j}$ where $A = \{a_i \mid i \in I\}$ where j ranges over some finite set of values, and so

$$\begin{aligned} fC &= \left(\prod_j (a_{n_j})^{n_j}\right) C \\ &= \prod_j \left((a_{i_j})^{n_j} C\right) = \prod_j (a_{i_j} C)^{n_j} \text{ be the definition of coset} \\ &\qquad\qquad\qquad\qquad \text{multiplication in } F[A]/C. \end{aligned}$$

So $F[A]/C$ is a free abelian group with basis $\{aC \mid a \in A\}$ (the representation of fC is unique since A is a generating set of $F[A]$—this follows from the definition of "reduced word" in $F[A]$). Here we have used multiplicative notation for the free abelian group, as opposed to the additive notation used in Section 38. By renaming the basis elements aC as a, we can view $F[A]/C$ as a free abelian group with basis A (of course, this "renaming" is accomplished with a mapping ϕ which is an isomorphism).

Note. The previous not indicates how a free abelian group can be constructed with a given basis (up to isomorphism). So in the event we start with a free abelian group G with basis X, then as guaranteed by by the Universal Quotient Group Property, G is isomorphic to a quotient group of a free group. Ths quotient group is $F[X]/C$.

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