A Student's Question: Why The Hell Am I In This Class?

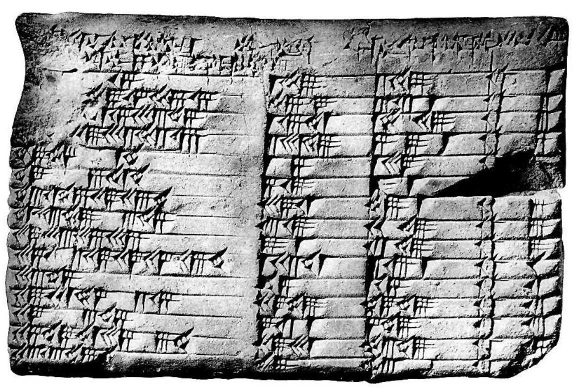
Note. Mathematics of the current era consists of the broad areas of (1) geometry, (2) analysis, (3) discrete math, and (4) algebra. This is an oversimplification; this is not a complete list and these areas are not disjoint. You are familiar with geometry from your high school experience, and analysis is basically the study of calculus in a rigorous/axiomatic way. You have probably encountered some of the topics from discrete math (graphs, networks, Latin squares, finite geometries, number theory). However, surprisingly, this is likely your first encounter with areas of algebra (in the modern sense). Modern algebra is roughly 100–200 years old, with most of the ideas originally developed in the nineteenth century and brought to rigorous completion in the twentieth century. Modern algebra is the study of groups, rings, and fields. However, these ideas grow out of the classical ideas of algebra from your previous experience (primarily, polynomial equations). The purpose of this presentation is to link the topics of classical algebra to the topics of modern algebra.

Babylonian Mathematics

The ancient city of Babylon was located in the southern part of Mesopotamia, about 50 miles south of present day Baghdad, Iraq. Clay tablets containing a type of writing called "cuneiform" survive from Babylonian times, and some of them reflect that the Babylonians had a sophisticated knowledge of certain mathematical ideas, some geometric and some arithmetic. [Bardi, page 28]

The best known surviving tablet with mathematical content is known as Plimpton 322. This tablet contains a list of Pythagorean triples, revealing some knowledge of the Pythagorean Theorem, as well as certain algebraic identities. This table also includes a list of ratios which would correspond to the cosecant of an angle of a right triangle determined by the triples. This feature of the tablet makes it the oldest known trigonometric table. This tablet is estimated to date from between 1900 and 1600 BCE. [Moar, page 8]

The Pythagorean triples were generated using the formula $(p^2 - q^2, 2pq, p^2 + q^2)$ where p and q are both positive integers, $p > q$, p and q are relatively prime, and exactly one of p and q is even. Therefore the Babylonians were aware of certain algebraic manipulations, though they would not have written any sort of formula as we do. [`http://www.math.ubc.ca/~cass/courses/m446-03/pl322/pl322.html`, accessed 12/29/2012]



Plimpton 322 (from the website mentioned above)

Another example of a Babylonian algebra problem [see Kleiner, page 1] is the following: "I have added the area and two-thirds of the side of my square and it is 0:35 [35/60 in sexagesimal notation]. What is the side of my square?" The solution is given verbally, as opposed to what we would consider an algebraic solution. In our notation, this problem can be stated as: "Solve for x where $x^2 + (2/3)x = 35/60$." The fact that the Babylonians could solve an equation of this form implies that they could solve any equations of the form $x^2 + ax = b$ where $a > 0$ and $b > 0$. This shows that the Babylonians were aware of the quadratic equation. Of course, none of this would be done using equations and the Babylonians would not admit negative numbers as solutions (or as numbers—numbers were thought of as *quantities* and so there was no meaning to a "negative quantity").

Egyptian Mathematics

Egyptian mathematics was centered more on practical, engineering-related problems than on abstraction. This is evidenced by the Rhind papyrus from 1650 BCE, which gives examples of problems that are basically arithmetical. Problem 21 asks for a solution to $\frac{2}{3} + \frac{1}{15} + x = 1$. Much of the content deals with addition of fractions of the form $1/n$. Again, the Egyptians did not use a notation or numerical symbols which we would recognize. [`http://www-history.mcs.st-and.ac.uk/HistTopics/Egyptian_papyri.html`]

Rhind Mathematical Papyus (from *Wikipedia*)

Greek Algebra

Euclid (from Wikipedia)

Euclid's *Elements of Geometry* (dating from about 300 BCE) is probably the most important single work in mathematics. It's axiom/theorem/proof style is used in virtually every advanced math textbook these days (including Fraleigh's). The *Elements* consist of 13 "books" (more appropriately, "chapters"). The first six books are on plane geometry (this is the basis of your high school geometry book) and the last three chapters cover solid geometry (read that as 3-dimensional geometry), climaxing in the proof that the five Platonic solids are the only regular solids. Books VII through X cover what might be called number theory. Euclid introduces the ideas of "greatest common divisor" and "least common multiple" (ideas we will encounter when studying finite groups and subgroups) and studies proportions, even/odd numbers, and incommensurability. We would relate the concept of incommensurability with irrational numbers. The Greeks were familiar with irrational numbers. You may be familiar with the famous story of the Pythagoreans' discovery that $\sqrt{2}$ is irrational (and the resulting fallout—this dates from about 550 BCE). Throughout the *Elements*, Euclid does not deal with numbers as we would think of them, but always with quantities—quantities of length, area, and volume. There is no concern with negatives, or even zero. [Artmann, pages 7–9]

Much of the work of the *Elements* concerns constructibility. For example, the first result in Book I is the construction of an equilateral triangle using a compass and straightedge (though the verbiage "compass and straightedge" is not used). In this spirit, there are the three classical construction problems of Greek mathematics: Squaring the circle, doubling the cube, and trisecting an angle. For two millenia, no solution was known for these constructions. Surprisingly, it has been shown that none of these three constructions can be accomplished (in a finite number of steps)! The proof of this is a result of modern algebra which dates from the 19th century.

Another very important work in the history of algebra is Diophantus's *Arithmetica* from about 250 CE. In it, Diophantus gave solutions to equations involving integers and rational numbers. He introduced a type of algebraic notation, gave rules for manipulating algebraic equations, and performed manipulations with negative numbers. In particular, he stated that "deficiency multiplied by deficiency yields availability" (that is, $(-a)(-b) = ab$). [Kleiner, page 3] In the 7th century, the Indian Brahmagupta gave this rule and several others relating to the arithmetic manipulation of negatives numbers. However, a wide acceptance of negative numbers was many centuries away. Another claim (by some) of fame for Brahmagupta is that he is the first to use zero as a number [Derbyshire, page 47].

The Arabic Numerals

Al-Khwarizmi (790–850) and Fibonacci (1170–1250)

(From MacTutor History of Mathematics)

The numerical symbols we are used to, 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9, are known as the Arabic numerals. However, they are in fact of Indian origin. They were carried through the Arabic world to Europe and this explains the fact that we know them as the "Arabic numerals." The Indian numerals were used in al-Khwarizmi's *ab al-jabr w'al-muqabala* (circa 800 CE), which "can be considered as the first book written on algebra." [MacTutor History of Math, al-Khwarizmi site] In fact, it is from the title of this book that we get our word "algebra." Al-Khwarizmi's name lead to our word "algorithm."

The "Arabic numerals" were used in Fibonacci's book *Liber abbaci* (published in 1202), and it is this book that spread the numerals through Europe. The first seven chapters of the book introduce the numerals and give examples on their use. The last eight chapters of the book include problems from arithmetic, algebra, and geometry. There are also problems relating to commerce. The use of a standardized numerical system by merchants further helped spread the numerals. [Derbyshire, page 68]

Details of this history are humorously told by Terry Jones (of "Monty Python") in the 2005 PBS documentary *The Story of 1*. This can be found on `YouTube` and `TopDocumentaryFilms.com`.

The Cubic and Quartic Equations

Tartaglia (1500–1557) and Cardano (1501–1576)

(From MacTutor History of Mathematics)

Before the year 1500, the only general polynomial equations which could be solved were linear equations $ax = b$ and quadratic equations $ax^2 + bx + c = 0$. The use of negative numbers was still not widespread. Around 1515, the Italian Scipione del Ferro was the first to give a solution to a (nontrivial) cubic equation of the form $ax^3 + bx = c$, though he never published the result. (Notice that this is rather impressive since, 500 years later, you are not familiar with a technique for solving such an equation [nor, offhand, am I]!) However, del Ferro did communicate the result to one of his students (Antonio Maria Fiore) who challenged Niccolò

Tartaglia to a public problem solving contest in 1535. Del Ferro only knew how to solve equations of the type given above, but Tartaglia knew how to solve many other types of cubic equations and easily won the contest (and the 16th century equivalent of tenure). Tartaglia reluctantly communicated his result to Gerolamo Cardano. Once he saw the solution, Cardano was able to find a proof for it. At this point (the late 1530s), Ludovico Ferrari, a secretary of Cardano's, learned of the work and was able to find a solution to the quartic equation in 1540 (Ferrari's solution involved a substitution that reduced the quartic equation to a cubic equation). In 1545, Cardano published *Ars Magna* ("*The Great Art*") in which he gave many details on the solutions of the cubic and quartic equations (Tartaglia became enraged at the publication of the cubic result, and this lead to a historical "battle" in the history of math between Tartaglia, del Ferro, and Cardano—a similar battle occurred about 150 years later over priority for the invention of calculus between Newton and Leibniz). The rapid discovery of a solution to the quartic equation following the cubic equation lead those involved to think that solutions of higher degree polynomial equations were on the horizon. Next, would be the quintic. [Derbyshire, pages 66–77]

Cardano presents dozens of cases for the solutions of cubic and quartic equations. This is due to the fact that negative numbers are still not accepted as "numbers." For example, Cardano would consider the cubic equations $x^3+2x=3$ and $x^3=4x+5$ to be from different "categories." Of course, both are of the form $ax^3+bx+c=0$ if we are allowed to use negative coefficients. So the notation used in the 16th century was not modern, but the solutions to the general equations were known.

For the sake of illustration, let's look at the solution to the cubic equation

$ax^3 + bx^2 + cx + d = 0$ in modern notation. The three solutions are:

$$x_1 = -\frac{b}{3a} - \frac{1}{3a}\sqrt[3]{\frac{1}{2}\left(2b^3 - 9abc + 27a^2d + \sqrt{(2b^3 - 9abc + 27a^2d^2)^2 - 4(b^2 - 3ac)^3}\right)}$$
$$-\frac{1}{3a}\sqrt[3]{\frac{1}{2}\left(2b^3 - 9abc + 27a^2d - \sqrt{(2b^3 - 9abc + 27a^2d^2)^2 - 4(b^2 - 3ac)^3}\right)}$$
$$x_2 = -\frac{b}{3a} + \frac{1+\sqrt{-3}}{6a}\sqrt[3]{\frac{1}{2}\left(2b^3 - 9abc + 27a^2d + \sqrt{(2b^3 - 9abc + 27a^2d^2)^2 - 4(b^2 - 3ac)^3}\right)}$$
$$+\frac{1-\sqrt{-3}}{6a}\sqrt[3]{\frac{1}{2}\left(2b^3 - 9abc + 27a^2d - \sqrt{(2b^3 - 9abc + 27a^2d^2)^2 - 4(b^2 - 3ac)^3}\right)}$$
$$x_3 = -\frac{b}{3a} + \frac{1-\sqrt{-3}}{6a}\sqrt[3]{\frac{1}{2}\left(2b^3 - 9abc + 27a^2d + \sqrt{(2b^3 - 9abc + 27a^2d^2)^2 - 4(b^2 - 3ac)^3}\right)}$$
$$+\frac{1+\sqrt{-3}}{6a}\sqrt[3]{\frac{1}{2}\left(2b^3 - 9abc + 27a^2d - \sqrt{(2b^3 - 9abc + 27a^2d^2)^2 - 4(b^2 - 3ac)^3}\right)}$$

This should explain why you are not familiar with this result!

Of particular historical interest to me (my Ph.D. area of study was complex analysis) is the impact these equations have had on the acceptance of complex numbers ("imaginary numbers," if you will). If we consider the cubic equation $x^3 - 15x - 4 = 0$ [Kleiner, page 7] then we find from the above equations that one solution is $x = \sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}}$. This equation can be manipulated by the "usual" algebraic rules with disregard for the fact that $\sqrt{-121}$ makes no sense by 16th century standards. The expression then reduces to $x = 4$ (notice $(4)^3 - 15(4) - 4 = 64 - 60 - 4 = 0$). So the equations above give a meaningful positive solution, even though computation of the solution involves the use of square roots of negatives. This application is where complex numbers gained a hold and eventually became a standard part of "numbers" and mathematics (though not until the 19th century, greatly motivated by Gauss's work). In fact, it is also as solutions to algebraic equations where negative numbers initially gained acceptance.

Unsolvability of the Quintic


Abel (1802–1829) and Galois (1811–1832)

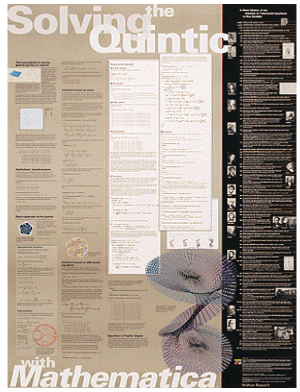
(From MacTutor History of Mathematics)

Between around 1550 and 1800, there were a number of mathematicians working on solving polynomial equations of degree 5. Prominent names are Rafael Bombelli (Italian), François Viéte (French), James Gregory (Scottish), Ehrenfried Walther von Tschirnhaus (German), Étienne Bézout (French), Leonhard Euler (Switzerland), Erland Samuel Bring (Sweden), and Joseph-Louis Lagrange (French). [Derbyshire, pages 79–83]

In 1799 Italian Paola Ruffini published a two volume work titled *General Theory of Equations* in which he included a "proof" that the quintic could not be algebraically solved. The proof ran 516 pages [Derbyshire, pages 87 and 88]. However, Ruffini's proof has been judged incomplete. The problem was that Ruffini lacked sufficient knowledge of "field theory," a topic initially developed in the early 19th century and a topic we will touch on in this class [Kleiner, page 63].

A correct proof that the quintic cannot be algebraically solved was given by the Norwegian Niels Henrik Abel in 1821 (Abel was not aware of Ruffini's alleged proof). Abel was plagued by poverty and in order to save money, he published his result in French in a six page pamphlet which was not widely circulated. Abel died in poverty in 1829 [Derbyshire, pages 96–99]. His work has been expanded and he is now viewed as one of the founders of modern algebra. One of the main structures we will study in this class is abelian groups, which are so-named after Abel. To add appreciation to the depth of Abel's result, observe that it is stated and proved in Theorem 56.6 on page 474 of our text—it is called the "*final goal*" and is the last result in the book!

We should elaborate on what is meant by "algebraically solve" an equation. This means that solutions can be found using the four arithmetic operations (addition/subtraction, multiplication/division) and the extraction of roots (square roots, cube roots, etc.). Notice that the quadratic equation involves only arithmetic operations and square roots; the cubic equations involve arithmetic operations, square roots, and cube roots. Examples of non-algebraic operations include logarithms/exponentials trigonometric functions, and series. What Abel showed was that there is no *algebraic* way to solve (in general) quintic equations—he did *not* show that there is no way to solve a quintic equation. In fact, the Wolfram software company (the developers of *Mathematica*) have a poster which explains how to *analytically* solve a quintic (that is, to precisely find the zeros using limits of infinite sequences).



Mathematica Quintic Poster (from
`http://library.wolfram.com/examples/quintic/`)

Another tragic figure from the history of algebra is the Frenchman Évariste Galois. Galois was born in 1811 and died in a dual in 1832 at the age of 20. He published five papers in 1829–30 (two appearing after his death). Galois gave the conditions under which a polynomial equation $p(x) = 0$ can be algebraically solved. In modern terms, he proved that $p(x) = 0$ can be algebraically solved (also called "solved be radicals") if and only if the group of $p(x)$ is solvable. Solvable groups are defined in Section VII.35 of our text, but Galois' main result belongs in the realm of a graduate-level field theory class. However, Part X of our text does touch on the topic of "Galois Theory." In fact, it is Galois theory which allows us to show that the three classical Greek constructions mentioned above cannot be accomplished. This is covered in our text in Section VI.32, following a discussion of constructible numbers.

The mathematical community was slow to accept Galois' result. In 1846, Joseph Liouville published the result, but it only became widely known in the 1870s, following Camille Jordan's publication of Galois' result (expanded and updated) in *Traité des substitutions et des équations algebraique.* Today, Galois Theory is a large area of modern mathematics (the American Mathematical Society even includes Galois Theory as a distinct area of mathematics, which they encode as "11R32"). For more historical details on Galois and his life, see `http://faculty.etsu.edu/gardnerr/Galois/Galois200.htm` (this is a website and presentation I prepared for the bicentennial of Galois' birth).

In a real sense, Galois, along with Abel, are the ones who gave birth to the modern algebra we study as undergraduates and graduates. Their work on polynomial equations from classical algebra lead to the study of the areas of groups, rings, fields, and extension fields. **That is why you are in this class!**

The Fundamental Theorem of Algebra

Descartes (1596–1650) and Gauss (1777–1855)

(From MacTutor History of Mathematics)

René Descartes in his *La géométie* (1637) proved the *Factor Theorem*: Polynomial $p(x)$ has $x = a$ as a zero (that is, $p(a) = 0$) if and only if $(x - a)$ is a factor of

$p(x)$. The Fundamental Theorem of Algebra concerns the zeros of a polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0.$$

It can be stated in a number of ways. One way is that every polynomial with complex coefficients has a complex zero. By the Factor Theorem, this implies that an n degree polynomial with complex coefficients can be factored into n (not necessarily distinct) linear factors. In terms of real polynomials, the Fundamental Theorem of Algebra can be stated as: Every polynomial with real coefficients can be written as a product of linear and quadratic polynomials with real coefficients.

The first proof of the Fundamental Theorem was given by d'Alembert in 1746, and a second proof was given shortly afterward by Euler. Both were incomplete and lacked rigor. Carl Friedrich Gauss gave several proofs, the first in his 1797 doctoral dissertation. Surprisingly, there is no purely algebraic proof of the Fundamental Theorem of *Algebra*! All known proofs require some result from *analysis*. The most common analytic result used is that an odd degree polynomial with real coefficients has a real zero (this can be shown using the Intermediate Value Theorem) [Kleiner, page 12]. Our text gives a proof of the Fundamental Theorem that uses Liouville's Theorem which involves analytic functions of a complex variable (see page 288 of Fraleigh—the proof is on a few lines long).

A Personal Voyage:[1] Why The Hell is Dr. Bob In This Class?

I took this class 30 years ago in Winter quarter 1983 at Auburn University in Montgomery. My instructor, Shirley A. Wilson, did an excellent job and I thought I might even want to pursue algebra in graduate school. I graduated from AUM in Spring 1984, and started the math graduate program at Auburn University in Fall 1984. My first year at Auburn, I completed the algebra sequence which covered Group Theory (Fall 1984), Ring Theory (Winter 1985), and Field Theory (Spring 1985). Bluntly put, I was traumatized by the graduate algebra sequence! It was horribly disorganized and virtually unintelligible. I managed to hack together a few concepts and learned a bit about field theory. As a consequence, my interest in algebra mostly dissolved. However, I did spend three years on my thesis topic, automorphisms of Steiner triple systems. This could be (creatively) classified as algebraic design theory. I finished this work in Summer 1987.

In Fall 1987 I started Ph.D. work in complex analysis. Though I still had a fear of algebra, my dissertation topic was partially related to algebraic topics. Part of my dissertation dealt with the zeros of polynomials (namely, the location of the zeros of a polynomial in the complex plane in terms of the coefficients of the polynomial). The rest of my dissertation dealt with more analytic results concerning properties of polynomials and other analytic functions. I finished my Ph.D. work in summer 1991.

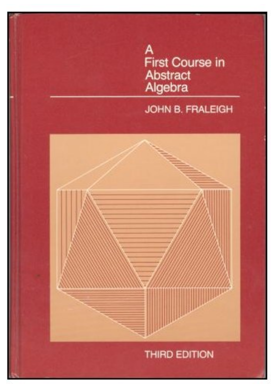
After two years at Louisiana State University in Shreveport during which I spent

[1]With apologies to Carl Sagan and the subtitle of *Cosmos* (©1980).

lots of time strengthening my publication record, I came to East Tennessee State University in Fall 1993. I was hired as an analyst and have spent the past 20 years teaching Analysis 1 and 2 (MATH 4217/5217 and 4227/5227), Real Analysis 1 and 2 (MATH 5210 and 5220), and Complex Analysis 1 and 2 (MATH 5510 and 5520). My attraction to analysis is based on its great geometric properties! That is, I can almost always draw a picture to illustrate a result in analysis. On the other hand, algebraic results are extremely abstract and it is very difficult (for me, at least) to draw pictures which illustrate the ideas of algebra. As a consequence, my analysis lectures are often none-to-charitable to results from algebra!

None-the-less, I am ready to face my fears and re-establish an old, valued relationship! In addition, I am ready to fill a large hole in my current knowledge: Introductory Modern Algebra. **That is why I am in this class!**

This is the background I bring to this class. As a result, I have a level of enthusiasm (and caution) which other instructors of this class may not share! I hope it proves useful in my approach to the instruction of this course.



The version of Fraleigh's text which I used as an undergraduate

(from `amazon.com`)

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Derbyshire, John. *Unknown Quantity: A Real and Imaginary History of Algebra*. Joseph Henry Press: 2006.

Kleiner, Israel. *A History of Abstract Algebra*. Birkhäuser: 2007.

The MacTutor History of Mathematics archive (of Saint Andrews University, Scotland), `http://www-history.mcs.st-and.ac.uk/`

Moar, Eli. *The Pythagorean Theorem: A 4,000-Year History*. Princeton University Press: 2007.

If you want to explore the history of algebra, then you might find of particular interest the Derbyshire book, the Kleiner book (which is rather short), and especially the MacTutor website. Three other readable algebra histories are:

Livio, Mario. *The Equation That Couldn't Be Solved*. Simon & Schuster: 2006.

Ronan, Mark. *Symmetry and the Monster: The Story of One of the Greatest Quests of Mathematics*. Oxford University Press: 2007.

Stewart, Ian. *Why Beauty Is Truth—A History of Symmetry*. Basic Books: 2007.

Two more technical works which concentrate on Abel and Galois individually are:

Pesic, Peter. *Abel's Proof: An Essay on the Sources and Meaning of Mathematical Unsolvability*. MIT Press: 2004.

Rigatelli, Toti (translated from Italian by Jon Denton). *Evariste Galois 1811–1832*. Birkhäser Verlag: 1996.

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