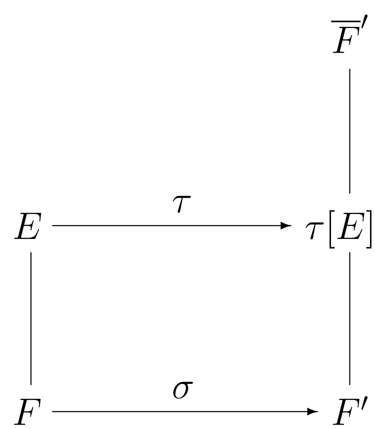
Section X.49. The Isomorphism Extension Theorem

Note. In this section we state and prove the Isomorphism Extension Theorem, the most important implication of which is the fact that the algebraic closure of a field is unique (up to isomorphism). We define the index of an extension field over a field and see that it behaves similarly to the index of a group over a subgroup.

Theorem 43.9. Isomorphism Extension Theorem.
Let E be an algebraic extension of a field F. Let σ be an isomorphism of F onto a field F'. Let $\overline{F}'$ be an algebraic closure of F'. Then σ can be extended to an isomorphism τ of E onto a subfield $\overline{F}'$ such that $\tau(a) = \sigma(a)$ for all $a \in F$.

Note. We give a proof at the end of this section. A diagram of the fields and isomorphisms is as follows:



Note. A special case of the Isomorphism Extension Theorem is the following which is the special case where σ is the conjugation isomorphism $\psi_{\alpha,\beta}$ of the previous section.

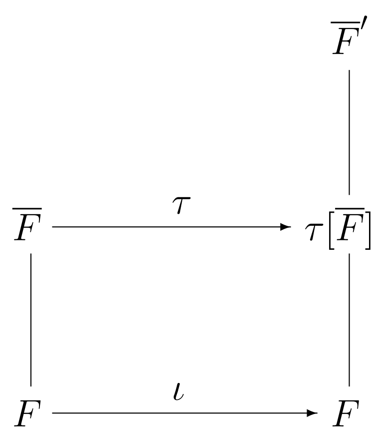
Corollary 49.4. If $E \leq \overline{F}$ is an algebraic extension of F and $\alpha, \beta \in E$ are conjugate over F, then the conjugation isomorphism $\psi_{\alpha,\beta} : F(\alpha) \to F(\beta)$, given by Theorem 48.3 can be extended to an isomorphism of E onto a subfield of $\overline{F}$.

Proof. This follows from the Isomorphism Extension Theorem by replacing F by $F(\alpha)$, F' by $F(\beta)$, $\overline{F}'$ by $\overline{F}$, and σ by $\psi_{\alpha,\beta}$. The extension of $\psi_{\alpha,\beta}$ is then τ as given in the Isomorphism Extension Theorem. ∎

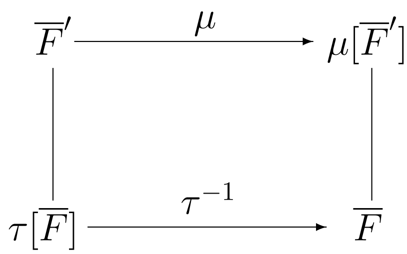
Note. The following result completes our exploration of the algebraic closure of a field.

Corollary 49.5. Let $\overline{F}$ and $\overline{F}'$ be two algebraic closures of F. Then $\overline{F}$ is isomorphic to $\overline{F}'$ under an isomorphism leaving each element of F fixed.

Proof. We use the Isomorphism Extension Theorem with E replaced by $\overline{F}$, F' replaced by F, and σ replaced with the identity ι.



Next, consider $\tau^{-1} : \tau[\overline{F}] \to \overline{F}$. Since τ is an isomorphism of $\overline{F}$ with $\tau[\overline{F}]$, then τ^{-1} is an isomorphism of $\tau[\overline{F}]$ with $\overline{F}$. So by the Isomorphism Extension Theorem τ^{-1} can be extended to an isomorphism of $\overline{F}'$ onto a subfield of $\overline{F}$. But τ^{-1} is already onto $\overline{F}$, then $\tau^{-1}[\overline{F}'] = \overline{F}$. So $\overline{F}$ and $\overline{F}'$ are isomorphic.



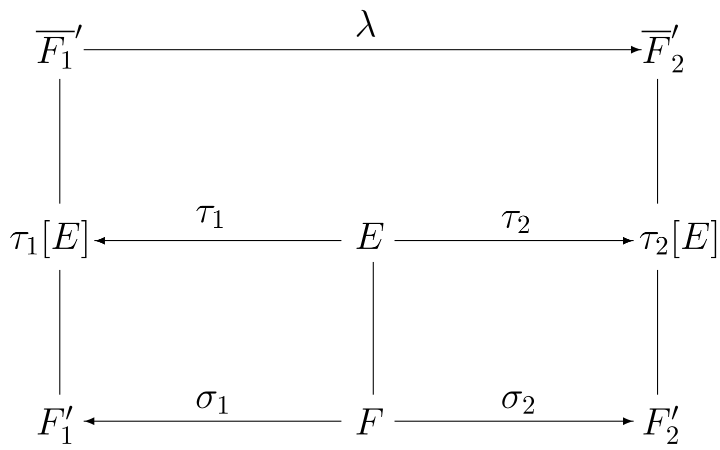
$$\Longrightarrow \quad \overline{F}' = \tau[\overline{F}] \xrightarrow{\mu = \tau^{-1}} \overline{F}$$

∎

Note. For E a finite extension of field F, we are interested in how many isomorphisms there are between E and a subfield of $\overline{F}$ which fix F. The following is a first step in this direction.

Theorem 49.7. Let E be a finite extension of field F. Let σ be an isomorphism of F onto a field F', and let $\overline{F}'$ be an algebraic closure of F'. Then the number of extensions of σ to an isomorphism τ of E onto a subfield of $\overline{F}'$ is finite, and independent of F', $\overline{F}'$, and σ. That is, the number of extensions is completely determined by the two fields E and F.

Note. We appeal to the following diagram in the proof of Theorem 49.7.

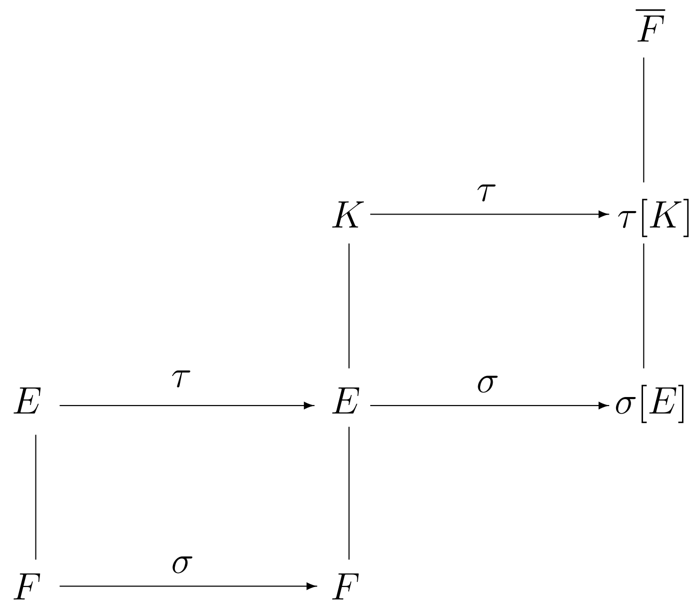


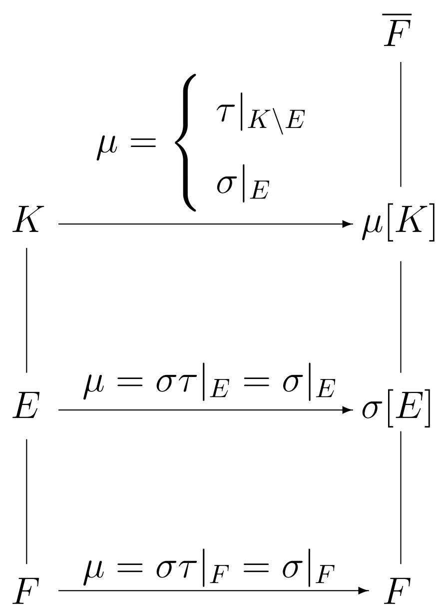
Definition 49.9. Let E be a finite extension of a field F. The number of isomorphisms of E onto a subfield of $\overline{F}$ leaving F fixed (which is finite by Theorem 49.7) is the *index of E over F*, denoted $\{E : F\}$.

Note. The following result shows that the index of a field behave similarly to the index of groups (compare to Theorem 10.14).

Corollary 49.10. If $F \leq E \leq K$ where K is a finite extension field of the field F, then $\{K : F\} = \{K : E\}\{E : F\}$.

Note. The diagrams illustrating the proof of Corollary 49.10 are:



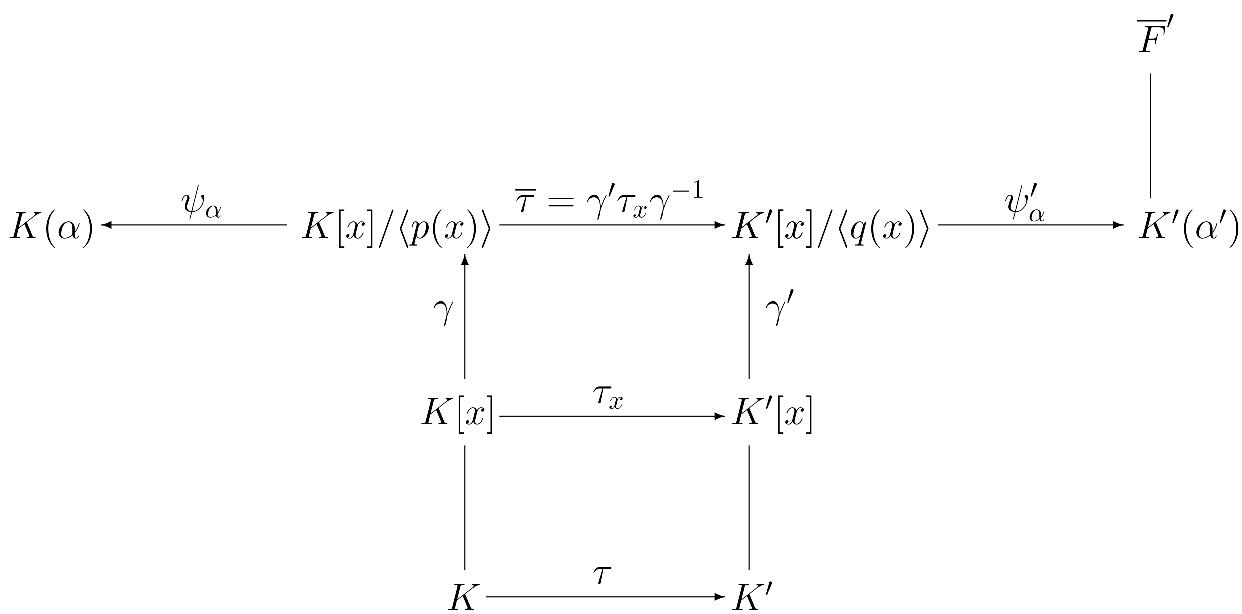


Example 49.11. To illustrate Corollary 49.10, consider again $\mathbb{Q}(\sqrt{2}, \sqrt{3})$. We saw in Example 48.17 that $\{\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}\} = 4$ (based on mapping extension elements $\sqrt{2}$ to $\sqrt{3}$ to their conjugates). Also, $\{\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})\} = 2$ and $\{\mathbb{Q}(\sqrt{2}) : \mathbb{Q}\} = 2$ (again by considering conjugates). So we have

$$4 = \{\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}\} = \{\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})\}\{\mathbb{Q}(\sqrt{2}) : \mathbb{Q}\} = 2 \cdot 2 = 4.$$

Note. Now for the lengthy proof of the Isomorphism Extension Theorem.

Note. The following diagram gives more details on the mappings in the proof.



The diagram includes $\tau_x : K[x] \to K'[x]$ which is an extension of τ from K to $K[x]$ (where the elements of K are treated as constants in $K[x]$), so

$$\tau_x(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) = \tau(a_0) + \tau(a_1)x + \tau(a_2)x^2 + \cdots + \tau(a_n)x^n.$$

The mappings $\gamma : K[x] \to K[x]/\langle p(x)\rangle$ and $\gamma' : K'[x] \to K'[x]/\langle q(x)\rangle$ are the canonical homomorphisms (for example, for $r(x) \in K[x]$ we have $\gamma(r(x)) = r(x) + \langle p(x)\rangle$).

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