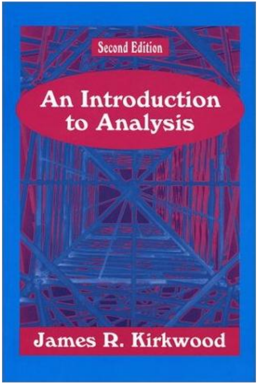
Analysis 1

Chapter 1. The Real Number System

1-1. Sets and Functions—Proofs of Theorems



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Theorem 1-1(a)

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(a) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$,

(b) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

Proof. (a) We show $A \setminus (B \cup C) \subset (A \setminus B) \cap (A \setminus C)$ and $(A \setminus B) \cap (A \setminus C) \subset A \setminus (B \cup C)$. Equality of the sets then holds by Note 1.1.C.

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Let $x \in A \setminus (B \cup C)$. Then $x \in A$ and $x \notin B \cup C$. Since $x \notin B \cup C$, then $x \notin B$ and $X \notin C$. So $x \in A \setminus B$ and $x \in A \setminus C$; that is $x \in (A \setminus B) \cap (A \setminus C)$. Therefore, $A \setminus (B \cup C) \subset (A \setminus B) \cap (A \setminus C)$

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Let $x \in A \setminus (B \cup C)$. Then $x \in A$ and $x \not\in B \cup C$. Since $x \not\in B \cup C$, then $x \not\in B$ and $X \not\in C$. So $x \in A \setminus B$ and $x \in A \setminus C$; that is $x \in (A \setminus B) \cap (A \setminus C)$. Therefore, $A \setminus (B \cup C) \subset (A \setminus B) \cap (A \setminus C)$

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Corollary 1-1(a)

Corollary 1-1. DeMorgan's Laws.
If B and C are sets (with universal set A) then

(a) $(B \cup C)^c = B^c \cap C^c$,
(b) $(B \cap C)^c = B^c \cup C^c$.

Proof. (a) Let A denote the universal set. Then $A \setminus (B \cup C) = A \cap (B \cup C)^c = (B \cup C)^c$. By Theorem 1-1(a),

$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C). \qquad (*)$$

But, again since A is the universal set, $A \setminus (B \cup C) = (B \cup C)^c$, $A \setminus B = B^c$, and $A \setminus C = C^c$. So by $(*)$, $(B \cup C)^c = B^c \cap C^c$, as claimed. $\square$

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Theorem 1-2(a)

Theorem 1-2. Suppose $f : A \to B$ and $g : B \to C$, and $g \circ f$ exists.

(a) If f and g are one-to-one, then $g \circ f$ is one-to-one.

(b) If f and g are onto, then $g \circ f$ is onto.

Proof. (a) Suppose $x_1 \neq x_2$ are arbitrary elements of the domain of f. Then $(g \circ f)(x_1) = g(f(x_1)) = g(y_1)$ where $y_1 = f(x_1)$. Also $(g \circ f)(x_2) = g(f(x_2)) = g(y_2)$ where $y_2 = f(x_2)$. Now $y_1 \neq y_2$ since f is hypothesized to be one-to-one. Since g is hypothesized to be one-to-one, $g(y_1) \neq g(y_2)$. That is, $(g \circ f)(x_1) \neq (g \circ f)(x_2)$ and, since x_1 and x_2 are arbitrary elements in the domain of f, then $g \circ f$ is one-to-one, as claimed. □

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Exercise 1.1.13(d)

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Proof. Let $x \in f^{-1}(B_1 \cap B_2)$. Then there exists $y \in B_1 \cap B_2$ such that $y = f(x)$. So $y \in B_1$ and $y \in B_2$, which implies $x \in f^{-1}(B_1)$ and $x = f^{-1}(B_2)$; that is, $x \in f^{-1}(B_1) \cap f^{-1}(B_2)$. So $f^{-1}(B_1 \cap B_2) \subset f^{-1}(B_1) \cap f^{-1}(B_2)$.

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