Analysis 1

Chapter 1. The Real Number System

1-2. Properties of the Real Numbers as an Ordered Field—Proofs of Theorems

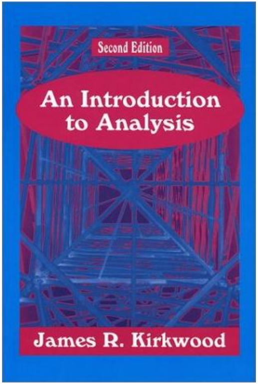

Table of contents

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Suppose both 0 and $\overline{0}$ are additive identities for field $\mathbb{F}$. Then $\overline{0} = 0 = \overline{0}$ since 0 is an additive identity. Also, $0 + \overline{0} = 0$ since $\overline{0}$ is an additive identity. Therefore

$$\begin{aligned} \overline{0} &= \overline{0} + 0 \\ &= 0 + \overline{0} \text{ since addition is commutative} \\ &\quad \text{ in a field by property (3)} \\ &= 0. \end{aligned}$$

That is, $0 = \overline{0}$ and the additive identity is unique, as claimed. □

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Theorem 1.4

Theorem 1-4. For $\mathbb{F}$ a field and $a \in \mathbb{F}$, the additive and multiplicative inverses of a are unique.

Proof. We give the proof for the multiplicative inverse and leave the proof for the additive inverse to Exercise 2.1.1.

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Proof. We give the proof for the multiplicative inverse and leave the proof for the additive inverse to Exercise 2.1.1.

Let $a \in \mathbb{F}$, $a \neq 0$, and suppose both b and $\overline{b}$ are multiplicative inverses of a. Then $a \cdot b = b \cdot a = 1$ and $a \cdot \overline{b} = \overline{b} \cdot a = 1$. So

$$\begin{aligned}
\overline{b} &= \overline{b} \cdot 1 \text{ since 1 is the multiplicative identity} \\
&= \overline{b} \cdot (a \cdot b) \text{ since } b \text{ is a multiplicative inverse of } a \\
&= (\overline{b} \cdot a) \cdot b \text{ since multiplication is associative by field property (2)} \\
&= 1 \cdot b \text{ since } \overline{b} \text{ is a multiplicative inverse of } a \\
&= b \quad \text{since 1 is the multiplicative identity.}
\end{aligned}$$

That is, $b = \overline{b}$ and the multiplicative inverse of a is unique. Since a is an arbitrary nonzero element of $\mathbb{F}$, then the claim follows. □

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Theorem 1-5

Theorem 1-5. For $\mathbb{F}$ a field, $a \cdot 0 = 0$ for all $a \in \mathbb{F}$.

Proof. Let $a \in \mathbb{F}$. Then $a + 0 = a$ by field property (5). So

$$\begin{aligned} a \cdot a &= a \cdot (a + 0) \\ &= a \cdot a + a \cdot 0 \text{ by field property (4).} \end{aligned}$$

Now $a \cdot a$ has an additive inverse by field property (6), denoted $-(a \cdot a)$, and adding this to both sides of the previous equation we have (by commutivity, field property (3))

$$a \cdot a + (-(a \cdot a)) = a \cdot a + a \cdot 0 + (-(a \cdot a)) = a \cdot a + (-(a \cdot a)) + a \cdot 0,$$

which implies $0 = 0 + a \cdot 0$ or $0 = a \cdot 0$ (by field property (5)), as claimed. □

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Theorem 1-6(c)

Theorem 1-6. For $\mathbb{F}$ a field and $a, b \in \mathbb{F}$:

(a) $a \cdot (-b) = (-a) \cdot b = -(a \cdot b)$.

(b) $-(-a) = a$.

(c) $(-a) \cdot (-b) = a \cdot b$.

Proof. **(c)** We have

$$\begin{aligned} 0 &= (-a) \cdot 0 \text{ Theorem 1-5} \\ &= (-a) \cdot (b + (-b)) \text{ by field property (6)} \\ &= (-a) \cdot b + (-a) \cdot (-b) \text{ by field property (4)} \end{aligned}$$

This shows that $(-a) \cdot b$ is an additive inverse of $(-a) \cdot (-b)$.

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This shows that $(-a) \cdot b$ is an additive inverse of $(-a) \cdot (-b)$. By part (a) $-(a \cdot b) = (-a) \cdot b$ so that $-(a \cdot b)$ is an additive inverse of $(-a) \cdot (-b)$ and, conversely, $(-a) \cdot (-b)$ is an additive inverse of $-(a \cdot b)$. But $a \cdot b$ is also an additive inverse of $-(a \cdot b)$. By Theorem 1-4 additive inverses are unique, so we must have $a \cdot b = (-a) \cdot (-b)$, as claimed. □

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Exercise 1.2.5

Exercise 1.2.5. If $\mathbb{F}$ is an ordered field, $a, b \in \mathbb{F}$ with $a \leq b$ and $b \leq a$ then $a = b$.

Proof. If $a \leq b$ then either $b - a \in P$ or $a = b$. If $b \leq a$ then either $a - b \in P$ or $a = b$. ASSUME $a \neq b$ (we give a proof by contradiction). Then it must be that both $b - a \in P$ and $a - b \in P$. But $(a - b) = -(b - a)$ since

$$\begin{aligned}(a-b)+(b-a) &= a-b+b-a = a+(-b+b)-a \text{ by associativity, field property (2)} \\ &= a+0-a = a-a = 0 \text{ by field properties (5) \& (6).}\end{aligned}$$

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But by the Law of Trichotomy, we cannot have both $(b - a) \in P$ and $(a - b) = -(b - a) \in P$ since these are nonzero additive inverses of each other, a CONTRADICTION. So the assumption that $a \neq b$ is false and we must have $a = b$, as claimed. □

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Theorem 1-7(b,c)

Theorem 1-7. Let $\mathbb{F}$ be an ordered field. For $a, b, c \in \mathbb{F}$:

(a) If $a < b$ then $a + c < b + c$.

(b) If $a < b$ and $b < c$ then $a < c$ ("$<$" is *transitive*).

(c) If $a < b$ and $c > 0$ then $ac < bc$.

(d) If $a < b$ and $c < 0$ then $ac > bc$.

(e) If $a \neq 0$ then $a^2 = a \cdot a > 0$.

Proof. (b) If $a < b$ and $b < c$ then, by definition of $<$, $b - a \in P$ and $c - b \in P$. Since P is closed under addition, $(b - a) + (c - b) = b - a + c - b = c - a \in P$ so that $a < c$ as claimed.

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(c) If $a < b$ and $c > 0$ then, by definition of $<$ and $>$, $b - a \in P$ and $c - 0 = c \in P$. Since P is closed under multiplication, then $(b - a)c \in P$. Now, by distribution (field property (4)), $(b - a)c = bc - ac$ so that $bc - ac \in P$ or $ac < bc$, as claimed.

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Exercise 1.2.7. Prove:

(a) $1 > 0$.

(b) If $0 < a < b$ then $0 < 1/b < 1/a$.

(c) If $0 < a < b$ then $a^n < b^n$ for natural number n.

(d) If $a > 0$, $b > 0$ and $a^n < b^n$ for some natural number n, then $a < b$.

(e) For any real numbers a and b, we have $|a| \leq |b|$ if and only if $a^2 \leq b^2$.

(f) Prove Theorem 1-10.

Proof. (a) By the Law of Trichotomy, either $1 < 0$ (in which case $0 - 1 = -1 \in P$), $1 > 0$ (in which case $1 - 0 = 1 \in P$), or $1 = 0$. Well, $1 \neq 0$ (this is part of the definition of a field; see property (5)). ASSUME $1 < 0$ so that $-1 \in P$. Then $(-1)(-1) = 1$ by Theorem 1-6(c) (we now start omitting the "$\cdot$" when multiplying). But if $-1 \in P$, then $(-1)(-1) = 1 \in P$, a CONTRADICTION to the Law of Trichotomy. So the assumption that $1 < 0$ is false and we must have $1 > 0$. □

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Exercise 1.2.7 (continued 1)

Exercise 1.2.7. Prove:

(b) If $0 < a < b$ then $0 < 1/b < 1/a$.

Proof (continued). (b) Suppose $0 < a < b$. Then $a, b, b - a \in P$. Consider $a^{-1} - b^{-1} = 1/a - 1/b$. We have

$$\begin{aligned}
(b-a)a^{-1}b^{-1} &= ba^{-1}b^{-1} - aa^{-1}b^{-1} \text{ by distribution, field property (4)} \\
&= bb^{-1}a^{-1} - aa^{-1}b^{-1} \text{ by commutivity, field property (3)} \\
&= 1a^{-1} - 1b^{-1} = a^{-1} - b^{-1} \text{ by field properties (5) \& (7)} \\
&= 1/a - 1/b.
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Now, for $a > 0$ we have $a^{-1} > 0$, for if not then we would have $a^{-1} < 0$ by the Law of Trichotomy and then $a(-a^{-1}) = -aa^{-1} = -1 \in P$ (since P is closed under multiplication), a contradiction to part (a). So both a^{-1} and b^{-1} are in P. Therefore, $(b-a)a^{-1}b^{-1} = 1/a - 1/b \in P$ (since P is closed under multiplication) and $0 < 1/b < 1/a$, as claimed. □

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Exercise 1.2.7 (continued 2)

Exercise 1.2.7. Prove:

(c) If $0 < a < b$ then $a^n < b^n$ for natural number n.

(d) If $a > 0$, $b > 0$ and $a^n < b^n$ for some natural number n, then $a < b$.

Proof (continued). (c) Suppose $0 < a < b$. Then by Theorem 1-7(c), $aa < ab$ and $ab < bb$, or $a^2 < ab < b^2$ and the result holds for $n = 1$ and $n = 2$. The general result holds by mathematical induction. □

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(d) ASSUME not. That is, suppose the hypotheses hold and ASSUME $a \geq b$. If $a = b$ then $a^n = b^n$ and we have a CONTRADICTION. If $a > b$ then $a^n > b^n$ by part (c), and we have a CONTRADICTION. So the assumption that $a \geq b$ cannot hold, and we must have $a < b$, as claimed. □

Exercise 1.2.7 (continued 2)

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Exercise 1.2.7 (continued 3)

Exercise 1.2.7. Prove:

(e) For any real numbers a and b, we have $|a| \leq |b|$ if and only if $a^2 \leq b^2$.

Proof (continued). (e) First, if $a = 0$ then $0 = |a| \leq |b|$ (by the definition of absolute value) for all b, and $a^2 = 0 \leq b^2$ for all b by Theorem 1-7(e). If $b = 0$ then $|a| \leq |b| = 0$ implies $a = 0$ (by the definition of absolute value), and $a^2 \leq b^2 = 0$ implies $a = 0$ by Theorem 1-7(e). So the claims hold if either $a = 0$ or $b = 0$, and we may now assume without loss of generality that a and b are both nonzero. Notice that for $a \neq 0$ we have $a^2 > 0$ by Theorem 1-7(e), and by Theorem 1-13(d) $|a^2| = |a|^2$, so that $a^2 = |a^2| = |a|^2$ when $a \neq 0$.

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Proof (continued). (e) First, if $a = 0$ then $0 = |a| \leq |b|$ (by the definition of absolute value) for all b, and $a^2 = 0 \leq b^2$ for all b by Theorem 1-7(e). If $b = 0$ then $|a| \leq |b| = 0$ implies $a = 0$ (by the definition of absolute value), and $a^2 \leq b^2 = 0$ implies $a = 0$ by Theorem 1-7(e). So the claims hold if either $a = 0$ or $b = 0$, and we may now assume without loss of generality that a and b are both nonzero. Notice that for $a \neq 0$ we have $a^2 > 0$ by Theorem 1-7(e), and by Theorem 1-13(d) $|a^2| = |a|^2$, so that $a^2 = |a^2| = |a|^2$ when $a \neq 0$.

Suppose $|a| \leq |b|$ By Theorem 1-7(c) we have $|a| \cdot |a| \leq |b| \cdot |a|$ and $|a| \cdot |b| \leq |b| \cdot |b|$, which implies $|a|^2 \leq |a| \cdot |b| \leq |b|^2$. As argued above, $a^2 = |a|^2$ and (similarly) $b^2 = |b|^2$, so we now have $a^2 \leq b^2$, as claimed.

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Proof (continued). (e) First, if $a = 0$ then $0 = |a| \leq |b|$ (by the definition of absolute value) for all b, and $a^2 = 0 \leq b^2$ for all b by Theorem 1-7(e). If $b = 0$ then $|a| \leq |b| = 0$ implies $a = 0$ (by the definition of absolute value), and $a^2 \leq b^2 = 0$ implies $a = 0$ by Theorem 1-7(e). So the claims hold if either $a = 0$ or $b = 0$, and we may now assume without loss of generality that a and b are both nonzero. Notice that for $a \neq 0$ we have $a^2 > 0$ by Theorem 1-7(e), and by Theorem 1-13(d) $|a^2| = |a|^2$, so that $a^2 = |a^2| = |a|^2$ when $a \neq 0$.

Suppose $|a| \leq |b|$ By Theorem 1-7(c) we have $|a| \cdot |a| \leq |b| \cdot |a|$ and $|a| \cdot |b| \leq |b| \cdot |b|$, which implies $|a|^2 \leq |a| \cdot |b| \leq |b|^2$. As argued above, $a^2 = |a|^2$ and (similarly) $b^2 = |b|^2$, so we now have $a^2 \leq b^2$, as claimed.

Exercise 1.2.7 (continued 4)

Exercise 1.2.7. Prove:

(e) For any real numbers a and b, we have $|a| \leq |b|$ if and only if $a^2 \leq b^2$.

(f) Prove Theorem 1-10: Suppose $0 < x < y$ are real numbers, and $s = p/q$ is a rational number. Then $x^s < y^s$.

Proof. (e) (continued). Suppose $a^2 \leq b^2$. Then $|a|^2 \leq |b|^2$ and, since $|a| > 0$ and $|b| > 0$, we have by part (d) with $n = 2$ that $|a| < |b|$, as claimed. □

(f) We have $x^s = (x^p)^{1/q}$ and $y^s = (y^p)^{1/q}$ for some $p, q \in \mathbb{Z}$. Since $0 < x < y$, then $x^p < y^q$ by part (c), and so

$$x^p = ((x^p)^{1/q})^q < ((y^p)^{1/q})^q = y^p.$$

Therefore by part (d), $x^p)^{1/q} < (y^p)^{1/q}$, or $x^s < y^s$, as claimed. □

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Theorem 1-11

Theorem 1-11. For $m, j \in \mathbb{N}$ with $j \leq m$ we have

$$\binom{m}{j} + \binom{m}{j-1} = \binom{m+1}{j}.$$

Proof. We have

$$\begin{aligned}
\binom{m}{j} + \binom{m}{j-1} &= \frac{m!}{j!(m-j)!} + \frac{m!}{(m-(j-1))!(j-1)!} \\
&= \frac{m!(m+1-j)}{j!(m-j)!(m+1-j)} + \frac{m!j}{(m+1-j)!(j-1)!j} \\
&= \frac{m!((m+1-j)+j)}{(m+1-j)!j!} = \frac{(m+1)!}{(m+1-j)!j!} = \binom{m+1}{j}.
\end{aligned}$$

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Theorem 1-12

Theorem 1-12. The Binomial Theorem.
Let a and b be real numbers and let $m \in \mathbb{N}$. Then

$$(a+b)^m = \sum_{j=0}^{m} \binom{m}{j} a^j b^{m-j}.$$

Proof. We give a proof based on mathematical induction. For the base case $m = 1$ we have

$$\sum_{j=0}^{1} \binom{1}{j} a^j b^{1-j} = \binom{1}{0} b + \binom{1}{1} a = b + a = (a+b)^1.$$

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For the induction hypothesis, suppose the claim holds for $m = k$ and that

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Consider the case $m = k + 1$.

Theorem 1-12 (continued 1)

Proof (continued). We have

$$\begin{aligned}
(a+b)^{k+1} &= (a+b)^k(a+b) = \left(\sum_{j=0}^{k}\binom{k}{j}a^j b^{k-j}\right)(a+b) \\
&\quad \text{by the induction hypothesis} \\
&= \sum_{j=0}^{k}\binom{k}{j}a^{j+1}b^{k-j} + \sum_{j=0}^{k}\binom{k}{j}a^j b^{k+1-j} \\
&= \sum_{\ell=1}^{k+1}\binom{k}{\ell-1}a^{\ell}b^{k-\ell+1} + b^{k+1} + \sum_{j=0}^{k}\binom{k}{j}a^j b^{k+1-j} \\
&\quad \text{where } \ell = j+1 \text{ so that } j = \ell - 1 \\
&= \sum_{\ell=1}^{k}\binom{k}{\ell-1}a^{\ell}b^{k-\ell+1} + a^{k+1} + b^{k+1} + \sum_{j=0}^{k}\binom{k}{j}a^j b^{k+1-j}
\end{aligned}$$

Theorem 1-12 (continued 2)

Proof (continued). We have

$$
\begin{aligned}
(a+b)^{k+1} &= a^{k+1}\sum_{j=1}^{k}\binom{k}{j-1}a^j b^{k+1-j} + \sum_{j=0}^{k}\binom{k}{j}a^j b^{k+1-j} + b^{k+1} \\
&\quad \text{replacing } \ell \text{ with } j \\
&= a^{k+1} + \sum_{j=1}^{k}\left(\binom{k}{j-1}+\binom{k}{j}\right)a^j b^{k+1-j} + b^{k+1} \\
&= a^{k+1} + \sum_{j=1}^{k}\binom{k+1}{j}a^j b^{k+1-j} \text{ by Theorem 1-11} \\
&= \sum_{j=0}^{k+1}\binom{k+1}{j}a^j b^{k+1-j}.
\end{aligned}
$$

Theorem 1-12 (continued 3)

Theorem 1-12. The Binomial Theorem.
Let a and b be real numbers and let $m \in \mathbb{N}$. Then

$$(a+b)^m = \sum_{j=0}^{m} \binom{m}{j} a^j b^{m-j}.$$

Proof (continued). The last equality holds because $a^{k+1} = \binom{k+1}{k+1} a^{k+1} b^0$ and $b^{k+1} = \binom{k+1}{0} a^0 b^{k+1}$. So the claim holds for $m = k+1$ and the induction step has been established. Therefore, by the Principle of Mathematical Induction, the claim holds for all $m \in \mathbb{N}$. □

Theorem 1-13(h)

Theorem 1-13. For all $a, b \in \mathbb{R}$

(g) $|a| < |b|$ if and only if $-b < a < b$.

(h) $|a + b| \leq |a| + |b|$ (this is the *Triangle Inequality*).

(i) $||a| - |b|| \leq |a - b|$.

Proof. (h) By part (c), $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$. So

$$-|a| - |b| \leq a + b \leq |a| + |b| \text{ by Exercise 1.2.4(a)}$$

or

$$-(|a| + |b|) \leq a + b \leq |a| + |b|,$$

so

$$|a + b| \leq |a| + |b| \text{ by part (g),}$$

as claimed. □

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$$|a + b| \leq |a| + |b| \text{ by part (g)},$$

as claimed. □