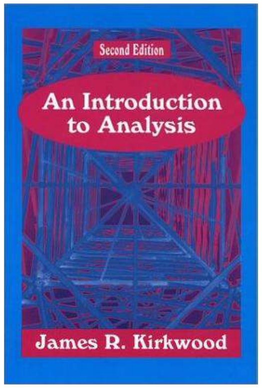
Analysis 1

Chapter 1. The Real Number System

1-3. The Completeness Axiom—Proofs of Theorems



Theorem 1-14

Theorem 1-14. If the lub and glb of a set of real numbers exists, then they are unique.

Proof. We show that the least upper bound is unique and leave the uniqueness of the greatest lower bound to Exercise 1.3.5.

Let A be a set of real numbers that is bounded above. Suppose α and $\overline{\alpha}$ are both least upper bounds of A. Then, by definition of least upper bound, α and $\overline{\alpha}$ are upper bounds of A and (since α is a *least* upper bound) then $\alpha \leq \overline{\alpha}$ and (since $\overline{\alpha}$ is a *least* upper bound) then $\overline{\alpha} \leq \alpha$. That is, $\alpha = \overline{\alpha}$ and so the least upper bound is unique. □

Theorem 1-15

Theorem 1-15.

(a) α is a lub of $A \subset \mathbb{R}$ if and only if

  (i) α is an upper bound of A, and

  (ii) For all $\varepsilon > 0$ there exists a number $x(\varepsilon) \in A$ such that $x(\varepsilon) > \alpha - \varepsilon$.

(b) β is a glb of $A \subset \mathbb{R}$ if and only if

  (i) β is a lower bound of A, and

  (ii) For all $\varepsilon > 0$ there exists a number $x(\varepsilon) \in A$ such that $x(\varepsilon) < \beta + \varepsilon$.

Proof. We give a proof of part (a) and leave the proof of part (b) to Exercise 1.3.3.

First, suppose $\alpha = \text{lub}(A)$. Then, by the definition of lub, α is an upper bound of A and so (i) holds. For (ii), let $\varepsilon > 0$ be arbitrary and given. Then $\alpha - \varepsilon < \alpha$ and so $\alpha - \varepsilon$ cannot be an upper bound for A since α is the least upper bound for A. Since $\alpha - \varepsilon$ is not an upper bound of set A then there is some element $x(\varepsilon) \in A$ with $x(\varepsilon) > \alpha - \varepsilon$, as claimed.

Theorem 1-15 (continued)

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(a) α is a lub of $A \subset \mathbb{R}$ if and only if

  (i) α is an upper bound of A, and

  (ii) For all $\varepsilon > 0$ there exists a number $x(\varepsilon) \in A$ such that $x(\varepsilon) > \alpha - \varepsilon$.

(b) β is a glb of $A \subset \mathbb{R}$ if and only if

  (i) β is a lower bound of A, and

  (ii) For all $\varepsilon > 0$ there exists a number $x(\varepsilon) \in A$ such that $x(\varepsilon) < \beta + \varepsilon$.

Proof (continued). Second, suppose (i) and (ii) hold. Then by (i), α is an upper bound of A. ASSUME $\overline{\alpha} < \alpha$ is also an upper bound of A. Let $\varepsilon = \alpha - \overline{\alpha} > 0$. Then $\overline{\alpha} = \alpha - \varepsilon$. By (ii), there is an element $x(\varepsilon) \in A$ with $x(\varepsilon) > \alpha - \varepsilon = \overline{\alpha}$. But this CONTRADICTS the hypothesis that $\overline{\alpha}$ is an upper bound of A. So the assumption that there is an upper bound of A less than α is false and so α is the least upper bound of A, as claimed. □

Theorem 1-16

Theorem 1-16. Let $\alpha = \text{lub}(A)$ and suppose $\alpha \notin A$. Then for all $\varepsilon > 0$, the interval $(\alpha - \varepsilon, \alpha)$ contains an infinite number of points of A.

Proof. We give a proof by contradiction. Let $\varepsilon > 0$ and let $\alpha = \text{lub}(A)$ where $\alpha \notin A$. By Theorem 1-15(a), there is a number $x(\varepsilon) \in A$ with $x(\varepsilon) > \alpha - \varepsilon$. Since $\alpha \notin A$ and α is an upper bound of A then $x(\varepsilon) \in (\alpha - \varepsilon, \alpha)$. ASSUME interval $(\alpha - \varepsilon, \alpha)$ contains only finitely many points of A, say $x_1 < x_2 < \ldots < x_n$. But then x_n is an upper bound of A and $x_n < \alpha$, CONTRADICTING the fact that α is the lub(A). So the assumption that interval $(\alpha - \varepsilon, \alpha)$ contains only finitely many points in A is false and hence $(\alpha - \varepsilon, \alpha)$ must contain infinitely many points of A, as claimed. □

Theorem 1-17(b)(i)

Theorem 1-17. Let A be a bounded set of real numbers, and suppose c is a real number. Then

(a) If $c > 0$: **(i)** $\text{lub}(cA) = c\,\text{lub}(A)$. **(ii)** $\text{glb}(cA) = c\,\text{glb}(A)$.

(b) If $c < 0$: **(i)** $\text{lub}(cA) = c\,\text{glb}(A)$. **(ii)** $\text{glb}(cA) = c\,\text{lub}(A)$.

Proof. Let $\alpha = \text{lub}(A)$ and $c < 0$. Then for $x \in A$ we have $x \le \alpha$. So $cx \ge c\alpha$. Therefore $c\alpha \le cx$ for any $x \in A$ and so $c\alpha$ is a lower bound for cA. To show $c\alpha$ is glb(A), let $\varepsilon > 0$. Then $\varepsilon/(-c) > 0$ and since $\alpha = \text{lub}(A)$ then by Theorem 1-15(a), there is element $x(\varepsilon) \in A$ such that $x(\varepsilon) > \alpha - \varepsilon/(-c)$. Then, since $c < 0$, $cx(\varepsilon) < c\alpha - c\varepsilon/(-c) = c\alpha + \varepsilon$. Since $cx(\varepsilon) \in cA$ then by Theorem 1-15(b), $c\alpha = \text{glb}(A)$, as claimed. □

Theorem 1-18. The Archimedean Principle

Theorem 1-18. The Archimedean Principle.
If $a, b \in \mathbb{R}$ and $a > 0$, then there is a natural number $n \in \mathbb{N}$ such that $na > b$.

Proof. Let $A = \{ka \mid n \in \mathbb{N}\}$. ASSUME A is bounded above. Then by the Axiom of Completeness, A has a least upper bound, say $\alpha = \text{lub}(A)$. Since $\alpha > 0$ there is an element of A, say Na, such that $\alpha - a < Na$. But then $\alpha < Na + a = (N+1)a$ (this is where we need α to be a *least* upper bound and not simply an upper bound) and $(N+1)a \in A$. This is a CONTRADICTION to the fact that α is an upper bound of A. So the assumption that A is bounded above is false and hence A has no upper bound. In particular, b is not an upper bound of A and so some element of A, say na, is greater than b, as claimed. □

Example 1.11

Example 1.11. Consider the set

$$A = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\} = \left\{\frac{n}{n+1} \,\middle|\, n \in \mathbb{N}\right\}.$$

Prove the least upper bound of A is 1.

Proof. Since $n < n+1$, then $\dfrac{n}{n+1} < 1$ by Theorem 1-7(c) (with $a = n$, $b = n+1$, and $c = 1/(n+1) > 0$). So 1 is an upper bound of A. Let $\varepsilon > 0$. With $a = \varepsilon$ and $b = 1$, we have by the Archimedean Principle (Theorem 1-18), that there is a positive integer N such that $N\varepsilon > 1$. Then $\varepsilon > 1/N$ by Theorem 1-7(c) (with $a = 1$, $b = N\varepsilon$, and $c = 1/N > 0$). Now we have

$$1 - \varepsilon < 1 - \frac{1}{N} = \frac{N-1}{N}.$$

Since $\frac{N-1}{N} \in A$, then by Theorem 1-15(a) 1 is the lub of A, as claimed. □

Exercise 1.3.4(a)

Exercise 1.3.4. (a) Between any two real numbers, there is a rational number. **(b)** Between any two real numbers, there is an irrational number.

Proof. We give a proof of (a) and leave part (b) as homework.

Let $a, b \in \mathbb{R}$, $a < b$. Then $b - a > 0$. By Corollary 1-18, we can find $N \in \mathbb{N}$ such that $1/N < b - a$. By the Archimedean Principle, there exists $k \in \mathbb{N}$ such that $k(1/N) > a$. Let K denote the smallest such k. Then

$$K(1/N) > a \geq (K-1)(1/N)$$

and

$$b = b - a + a > 1/N + a \geq 1/N + (K-1)(1/N) = K(1/N).$$

So, $K(1/N) \in (a, b)$ and of course $K/N \in \mathbb{Q}$. Therefore, between any two real numbers there is a rational number, as claimed. □

Theorem 1-19

Theorem 1-19. The union of a countable collection of countable sets is countable.

Proof. Since we have a countable collection of sets, denote the sets as E_1, $E_2, \ldots$. Since each set is countable, we can denote the elements of E_i as $x_{i1}, x_{i2}, \ldots$. Then

$$\cup_{i \in \mathbb{N}} E_i = \cup_{i \in \mathbb{N}} \left(\cup_{j \in \mathbb{N}} \{x_{ij}\} \right).$$

So let $f : \cup E_i \to \mathbb{N}$ as $f(x_{ij}) = 2^i 3^j$. Then f is one-to-one and so $\cup_{i \in \mathbb{N}} E_i$ is countable. □

Theorem 1-20

Theorem 1-20. The real numbers in $(0, 1)$ form an uncountable set.

Proof. Any real number in $(0, 1)$ can be uniquely represented in binary form as an infinite decimal (we use the usual binary representation if it is infinite and if a number is represented with a finite binary expansion, then we simply change the "last" 1 to a 0 and append an infinite number of 1's after this 0). ASSUME that these numbers are countable. Then let the set be $\{x_1, x_2, \ldots\}$ and suppose the binary representations are:

$$\begin{array}{ccccccc} x_1 & = & 0. & a_{11} & a_{12} & a_{13} & \cdots \\ x_2 & = & 0. & a_{21} & a_{22} & a_{23} & \cdots \\ x_3 & = & 0. & a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

where the a's give the binary representations of the x's.

Theorem 1-20 (continued)

Theorem 1-20. The real numbers in $(0, 1)$ form an uncountable set.

Proof (continued). Construct number $b = 0.\, b_1\, b_2\, b_3 \ldots$ where

$$b_i = \begin{cases} 0 \text{ if } a_{ii} = 1 \\ 1 \text{ if } a_{ii} = 0. \end{cases}$$

The $b \neq x_i$ for all i. Therefore, the list $x_1, x_2, x_2, \ldots$ does not include b, CONTRADICTING the assumption that the set of numbers in $(0, 1)$ is countable. So $(0, 1)$ is countable, as claimed. □

Theorem 1-21. Cantor's Theorem

Theorem 1-21. Cantor's Theorem.
The cardinal number of $\mathcal{P}(X)$ is strictly larger than the number of X.

Proof. First, notice that there is a one-to-one map $f : X \to \mathcal{P}(X)$, namely the function f mapping $x \mapsto \{x\}$. To show $|X| < |\mathcal{P}(X)|$, we must show that there is no onto function $f : X \to \mathcal{P}(X)$.

ASSUME $f : X \to \mathcal{P}(X)$ is an onto function. Notice that $f(x) \in \mathcal{P}(X)$ so $f(x)$ is itself a subset of X. So for some $x \in X$ we have $x \in f(x)$ and for others we have $x \notin f(x)$. Define set $A = \cup_{x \in X}\{x \mid x \notin f(x)\}$. Since f is onto (by assumption) then for some $a \in X$ we have $f(a) = A$. Now either $a \in A$ or $a \notin A$. If $a \in A$ then $a \notin f(a) = A$ by the definition of set A, a CONTRADICTION. If $a \notin A = f(a)$ then by the definition of set A we must have $a \in A$, another CONTRADICTION. Since one of these ($a \in A$ or $a \notin A$) must be the case, the assumption that there is an onto function from X to $\mathcal{P}(X)$ is false. So, by the definition, $|X| < |\mathcal{P}(X)|$. □

Exercise 1.3.9

Exercise 1.3.9. Let $y > 0$, $n \in \mathbb{N}$, and $A = \{x \mid x^n < y\}$.

(a) A is nonempty and bounded above.

(b) If $a \in \mathbb{R}$, $a > 0$, and $n \in \mathbb{N}$, then there exists $x \in \mathbb{R}$, $x > 0$, such that $x^n < a$. Use this to prove Theorem 1-8.

Proof. (a) Certainly $0 \in A$. If $y \le 1$, then 1 is an upper bound of A by Exercise 1.2.7(c). If $y > 1$ then y is an upper bound of A:

$$x^n < y \text{ implies } x^n < y^n \text{ implies } x < y \text{ by Exercise 1.2.8(b).}$$

In either case, A is bounded above, as claimed. □

(b) If $a < 1$ then there exists x such that $0 < x < a$ (either by the Archimedean Principle or by Corollary 1-18 with $b = 1$). Then $x^n < a < 1$ (by Exercise 1.2.8), and the claim follows. If $a > 1$ then let $x = 1$. Then $x^n = 1 < a$ and, again, the claim follows.

Exercise 1.3.9 (continued 1)

Proof (continued). Theorem 1-8 states: "Let $y \in \mathbb{R}^+$ and let $n \in \mathbb{N}$. Then there is a unique $z \in \mathbb{R}^+$ such that $z^n = y$."
Consider $\{x \mid x^n < y\} \subset \mathbb{R}$. This set is nonempty and bounded by Exercise 1.3.9(a), so it has a least upper bound by Axiom 9. Denote lub(A) as z. We need only show $z^n = y$. ASSUME $x^n \ne y$ and $z^n < y$. Let $y - z^n = \varepsilon > 0$, and so $x^z + \varepsilon = y$. Choose δ_i such that $\delta_i^{n-i} < \varepsilon\left\{n\binom{n}{i}z^i\right\}^{-1}$ for $i = 0, 1, 2, \ldots, n-1$. So

$$(z+\delta)^n = \sum_{i=0}^{n}\binom{n}{i}z^i\delta^{n-i} = z^n + \sum_{i=0}^{n-1}\binom{n}{i}z^i\delta^{n-i}$$

$$< z^n + \sum_{i=0}^{n-1}\binom{n}{i}z^i\left\{n\binom{n}{i}z^i\right\}^{-1}\varepsilon = z^n + \sum_{i=0}^{n-1}\frac{\varepsilon}{n} = z^n + \varepsilon = y.$$

So $z + \delta \in \{z \mid z^n < y\}$ where $\delta > 0$ and therefore z is not an upper bound of $\{x \mid x^n < y\}$, a CONTRADICTION.

Exercise 1.3.9 (continued 2)

Proof (continued). So the assumption that $z^n < y$ is false and we must have $z^n \ge y$.

Now ASSUME $x^n > y$. A CONTRADICTION will similarly follow by letting $z^n - y = \varepsilon$ and $y = z^n - \varepsilon$.

Uniqueness follows by assuming $a^n = y$. Then a^n is an upper bound of the above set. If a^n is not the least upper bound then $a^n > y$, so we must have $a = \text{lub}(A) = z$. That is, if $a^n = y$ then $a = z$ and the choice of z is unique. □

Exercise 1.3.10

Exercise 1.3.10. Let A be uncountable and B countable. Then $A \setminus B$ is uncountable.

Proof. We have $A \subset (A \setminus B) \cup B$. ASSUME the theorem is false, that $A \setminus B$ and B are countable and A is uncountable. But if $A \setminus B$ and B are countable, then $(A \setminus B) \cup B \supset A$ is countable by Theorem 1-19, a CONTRADICTION. So the assumption is false and hence set $A \setminus B$ is uncountable, as claimed. □