Analysis 1

Chapter 1. The Real Number System

1-3. The Completeness Axiom—Proofs of Theorems

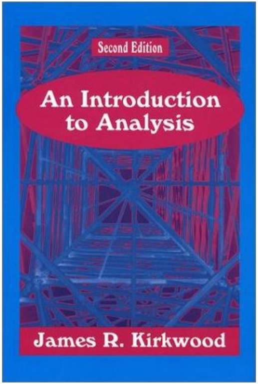

Table of contents

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Let A be a set of real numbers that is bounded above. Suppose α and $\overline{\alpha}$ are both least upper bounds of A. Then, by definition of least upper bound, α and $\overline{\alpha}$ are upper bounds of A and (since α is a *least* upper bound) then $\alpha \leq \overline{\alpha}$ and (since $\overline{\alpha}$ is a *least* upper bound) then $\overline{\alpha} \leq \alpha$. That is, $\alpha = \overline{\alpha}$ and so the least upper bound is unique. □

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Theorem 1-15

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(a) α is a lub of $A \subset \mathbb{R}$ if and only if
  (i) α is an upper bound of A, and
  (ii) For all $\varepsilon > 0$ there exists a number $x(\varepsilon) \in A$ such that $x(\varepsilon) > \alpha - \varepsilon$.

(b) β is a glb of $A \subset \mathbb{R}$ if and only if
  (i) β is a lower bound of A, and
  (ii) For all $\varepsilon > 0$ there exists a number $x(\varepsilon) \in A$ such that $x(\varepsilon) < \beta + \varepsilon$.

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Proof. We give a proof of part (a) and leave the proof of part (b) to Exercise 1.3.3.

First, suppose $\alpha = \text{lub}(A)$. Then, by the definition of lub, α is an upper bound of A and so (i) holds. For (ii), let $\varepsilon > 0$ be arbitrary and given. Then $\alpha - \varepsilon < \alpha$ and so $\alpha - \varepsilon$ cannot be an upper bound for A since α is the least upper bound for A. Since $\alpha - \varepsilon$ is not an upper bound of set A then there is some element $x(\varepsilon) \in A$ with $x(\varepsilon) > \alpha - \varepsilon$, as claimed.

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Proof (continued). Second, suppose (i) and (ii) hold. Then by (i), α is an upper bound of A. ASSUME $\overline{\alpha} < \alpha$ is also an upper bound of A. Let $\varepsilon = \alpha - \overline{\alpha} > 0$. Then $\overline{\alpha} = \alpha - \varepsilon$. By (ii), there is an element $x(\varepsilon) \in A$ with $x(\varepsilon) > \alpha - \varepsilon = \overline{\alpha}$. But this CONTRADICTS the hypothesis that $\overline{\alpha}$ is an upper bound of A. So the assumption that there is an upper bound of A less than α is false and so α is the least upper bound of A, as claimed. □

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Theorem 1-16. Let $\alpha = \text{lub}(A)$ and suppose $\alpha \notin A$. Then for all $\varepsilon > 0$, the interval $(\alpha - \varepsilon, \alpha)$ contains an infinite number of points of A.

Proof. We give a proof by contradiction. Let $\varepsilon > 0$ and let $\alpha = \text{lub}(A)$ where $\alpha \notin A$. By Theorem 1-15(a), there is a number $x(\varepsilon) \in A$ with $x(\varepsilon) > \alpha - \varepsilon$. Since $\alpha \notin A$ and α is an upper bound of A then $x(\varepsilon) \in (\alpha - \varepsilon, \alpha)$. ASSUME interval $(\alpha - \varepsilon, \alpha)$ contains only finitely many points of A, say $x_1 < x_2 < \ldots < x_n$. But then x_n is an upper bound of A and $x_n < \alpha$, CONTRADICTING the fact that α is the lub(A). So the assumption that interval $(\alpha - \varepsilon, \alpha)$ contains only finitely many points in A is false and hence $(\alpha - \varepsilon, \alpha)$ must contain infinitely many points of A, as claimed. □

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Theorem 1-17(b)(i)

Theorem 1-17. Let A be a bounded set of real numbers, and suppose c is a real number. Then

(a) If $c > 0$: **(i)** $\text{lub}(cA) = c\,\text{lub}(A)$. **(ii)** $\text{glb}(cA) = c\,\text{glb}(A)$.

(b) If $c < 0$: **(i)** $\text{lub}(cA) = c\,\text{glb}(A)$. **(ii)** $\text{glb}(cA) = c\,\text{lub}(A)$.

Proof. Let $\alpha = \text{lub}(A)$ and $c < 0$. Then for $x \in A$ we have $x \leq \alpha$. So $cx \geq c\alpha$. Therefore $c\alpha \leq cx$ for any $x \in A$ and so $c\alpha$ is a lower bound for cA. To show $c\alpha$ is $\text{glb}(A)$, let $\varepsilon > 0$.

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Theorem 1-18. The Archimedean Principle

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If $a, b \in \mathbb{R}$ and $a > 0$, then there is a natural number $n \in \mathbb{N}$ such that $na > b$.

Proof. Let $A = \{ka \mid n \in \mathbb{N}\}$. ASSUME A is bounded above. Then by the Axiom of Completeness, A has a least upper bound, say $\alpha = \text{lub}(A)$. Since $\alpha > 0$ there is an element of A, say Na, such that $\alpha - a < Na$. But then $\alpha < Na + a = (N+1)a$ (this is where we need α to be a *least* upper bound and not simply an upper bound) and $(N+1)a \in A$.

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Example 1.11

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$$A = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots \right\} = \left\{ \frac{n}{n+1} \,\middle|\, n \in \mathbb{N} \right\}.$$

Prove the least upper bound of A is 1.

Proof. Since $n < n+1$, then $\dfrac{n}{n+1} < 1$ by Theorem 1-7(c) (with $a = n$, $b = n+1$, and $c = 1/(n+1) > 0$). So 1 is an upper bound of A. Let $\varepsilon > 0$. With $a = \varepsilon$ and $b = 1$, we have by the Archimedean Principle (Theorem 1-18), that there is a positive integer N such that $N\varepsilon > 1$.

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$$1 - \varepsilon < 1 - \frac{1}{N} = \frac{N-1}{N}.$$

Since $\frac{N-1}{N} \in A$, then by Theorem 1-15(a) 1 is the lub of A, as claimed. □

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Exercise 1.3.4(a)

Exercise 1.3.4. (a) Between any two real numbers, there is a rational number. **(b)** Between any two real numbers, there is an irrational number.

Proof. We give a proof of (a) and leave part (b) as homework.

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Proof. We give a proof of (a) and leave part (b) as homework.

Let $a, b \in \mathbb{R}$, $a < b$. Then $b - a > 0$. By Corollary 1-18, we can find $N \in \mathbb{N}$ such that $1/N < b - a$. By the Archimedean Principle, there exists $k \in \mathbb{N}$ such that $k(1/N) > a$. Let K denote the smallest such k.

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$$K(1/N) > a \geq (K-1)(1/N)$$

and

$$b = b - a + a > 1/N + a \geq 1/N + (K-1)(1/N) = K(1/N).$$

So, $K(1/N) \in (a, b)$ and of course $K/N \in \mathbb{Q}$. Therefore, between any two real numbers there is a rational number, as claimed. □

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Theorem 1-19

Theorem 1-19. The union of a countable collection of countable sets is countable.

Proof. Since we have a countable collection of sets, denote the sets as E_1, E_2, ... Since each set is countable, we can denote the elements of E_i as $x_{i1}, x_{i2}, \ldots$. Then

$$\cup_{i\in\mathbb{N}} E_i = \cup_{i\in\mathbb{N}} \left(\cup_{j\in\mathbb{N}} \{x_{ij}\}\right).$$

So let $f : \cup E_i \to \mathbb{N}$ as $f(x_{ij}) = 2^i 3^j$. Then f is one-to-one and so $\cup_{i\in\mathbb{N}} E_i$ is countable. □

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Theorem 1-20

Theorem 1-20. The real numbers in $(0, 1)$ form an uncountable set.

Proof. Any real number in $(0, 1)$ can be uniquely represented in binary form as an infinite decimal (we use the usual binary representation if it is infinite and if a number is represented with a finite binary expansion, then we simply change the "last" 1 to a 0 and append an infinite number of 1's after this 0). ASSUME that these numbers are countable. Then let the set be $\{x_1, x_2, \ldots\}$ and suppose the binary representations are:

$$\begin{array}{ccccccc} x_1 & = & 0. & a_{11} & a_{12} & a_{13} & \cdots \\ x_2 & = & 0. & a_{21} & a_{22} & a_{23} & \cdots \\ x_3 & = & 0. & a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

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Theorem 1-20 (continued)

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Proof (continued). Construct number $b = 0.\, b_1\, b_2\, b_3\, \ldots$ where

$$b_i = \begin{cases} 0 \text{ if } a_{ii} = 1 \\ 1 \text{ if } a_{ii} = 0. \end{cases}$$

The $b \neq x_i$ for all i. Therefore, the list $x_1, x_2, x_2, \ldots$ does not include b, CONTRADICTING the assumption that the set of numbers in $(0, 1)$ is countable. So $(0, 1)$ is countable, as claimed. □

Theorem 1-21. Cantor's Theorem

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The cardinal number of $\mathcal{P}(X)$ is strictly larger than the number of X.

Proof. First, notice that there is a one-to-one map $f : X \to \mathcal{P}(X)$, namely the function f mapping $x \mapsto \{x\}$. To show $|X| < |\mathcal{P}(X)|$, we must show that there is no onto function $f : X \to \mathcal{P}(X)$.

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ASSUME $f : X \to \mathcal{P}(X)$ is an onto function. Notice that $f(x) \in \mathcal{P}(X)$ so $f(x)$ is itself a subset of X. So for some $x \in X$ we have $x \in f(x)$ and for others we have $x \notin f(x)$. Define set $A = \cup_{x \in X}\{x \mid x \notin f(x)\}$. Since f is onto (by assumption) then for some $a \in X$ we have $f(a) = A$.

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The cardinal number of $\mathcal{P}(X)$ is strictly larger than the number of X.

Proof. First, notice that there is a one-to-one map $f : X \to \mathcal{P}(X)$, namely the function f mapping $x \mapsto \{x\}$. To show $|X| < |\mathcal{P}(X)|$, we must show that there is no onto function $f : X \to \mathcal{P}(X)$.

ASSUME $f : X \to \mathcal{P}(X)$ is an onto function. Notice that $f(x) \in \mathcal{P}(X)$ so $f(x)$ is itself a subset of X. So for some $x \in X$ we have $x \in f(x)$ and for others we have $x \notin f(x)$. Define set $A = \cup_{x \in X}\{x \mid x \notin f(x)\}$. Since f is onto (by assumption) then for some $a \in X$ we have $f(a) = A$. Now either $a \in A$ or $a \notin A$. If $a \in A$ then $a \notin f(a) = A$ by the definition of set A, a CONTRADICTION. If $a \notin A = f(a)$ then by the definition of set A we must have $a \in A$, another CONTRADICTION. Since one of these ($a \in A$ or $a \notin A$) must be the case, the assumption that there is an onto function from X to $\mathcal{P}(X)$ is false. So, by the definition, $|X| < |\mathcal{P}(X)|$. □

Exercise 1.3.9

Exercise 1.3.9. Let $y > 0$, $n \in \mathbb{N}$, and $A = \{x \mid x^n < y\}$.

(a) A is nonempty and bounded above.

(b) If $a \in \mathbb{R}$, $a > 0$, and $n \in \mathbb{N}$, then there exists $x \in \mathbb{R}$, $x > 0$, such that $x^n < a$. Use this to prove Theorem 1-8.

Proof. (a) Certainly $0 \in A$. If $y \leq 1$, then 1 is an upper bound of A by Exercise 1.2.7(c). If $y > 1$ then y is an upper bound of A:

$$x^n < y \text{ implies } x^n < y^n \text{ implies } x < y \text{ by Exercise 1.2.8(b).}$$

In either case, A is bounded above, as claimed. □

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(b) If $a < 1$ then there exists x such that $0 < x < a$ (either by the Archimedean Principle or by Corollary 1-18 with $b = 1$). Then $x^n < a < 1$ (by Exercise 1.2.8), and the claim follows. If $a > 1$ then let $x = 1$. Then $x^n = 1 < a$ and, again, the claim follows.

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Exercise 1.3.9 (continued 1)

Proof (continued). Theorem 1-8 states: "Let $y \in \mathbb{R}^+$ and let $n \in \mathbb{N}$. Then there is a unique $z \in \mathbb{R}^+$ such that $z^n = y$."

Consider $\{x \mid x^n < y\} \subset \mathbb{R}$. This set is nonempty and bounded by Exercise 1.3.9(a), so it has a least upper bound by Axiom 9. Denote lub(A) as z. We need only show $z^n = y$. ASSUME $x^n \neq y$ and $z^n < y$. Let $y - z^n = \varepsilon > 0$, and so $x^z + \varepsilon = y$.

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$$(z+\delta)^n = \sum_{i=0}^{n} \binom{n}{i} z^i \delta^{n-i} = z^n + \sum_{i=0}^{n-1} \binom{n}{i} z^i \delta^{n-i}$$

$$< z^n + \sum_{i=0}^{n-1} \binom{n}{i} z^i \left\{ n \binom{n}{i} z^i \right\}^{-1} \varepsilon = z^n + \sum_{i=0}^{n-1} \frac{\varepsilon}{n} = z^n + \varepsilon = y.$$

So $z + \delta \in \{z \mid z^n < y\}$ where $\delta > 0$ and therefore z is not an upper bound of $\{x \mid x^n < y\}$, a CONTRADICTION.

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Exercise 1.3.9 (continued 2)

Proof (continued). So the assumption that $z^n < y$ is false and we must have $z^n \geq y$.

Now ASSUME $x^n > y$. A CONTRADICTION will similarly follow by letting $z^n - y = \varepsilon$ and $y = z^n - \varepsilon$.

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Uniqueness follows by assuming $a^n = y$. Then a^n is an upper bound of the above set. If a^n is not the least upper bound then $a^n > y$, so we must have $a = \text{lub}(A) = z$. That is, if $a^n = y$ then $a = z$ and the choice of z is unique. □

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Exercise 1.3.10

Exercise 1.3.10. Let A be uncountable and B countable. Then $A \setminus B$ is uncountable.

Proof. We have $A \subset (A \setminus B) \cup B$. ASSUME the theorem is false, that $A \setminus B$ and B are countable and A is uncountable. But if $A \setminus B$ and B are countable, then $(A \setminus B) \cup B \supset A$ is countable by Theorem 1-19, a CONTRADICTION. So the assumption is false and hence set $A \setminus B$ is uncountable, as claimed. □

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