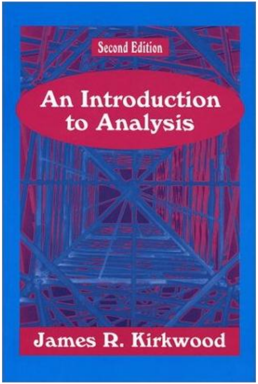
Analysis 1

Chapter 2. Sequences of Real Numbers

2-1. Sequences of Real Numbers—Proofs of Theorems



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Example 2.1.A

Example 2.1.A. Prove $\left\{\frac{1}{n}\right\}_{n=1}^{\infty} \to 0$.

Proof. Let $\varepsilon > 0$. By Corollary 1-18 (with $a = \varepsilon$ and $b = 1$), there is $N(\varepsilon) \in \mathbb{N}$ such that $1/N(\varepsilon) < \varepsilon$. For any $n > N(\varepsilon)$ we have

$$|x_n - L| = \left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| = \frac{1}{n} < \frac{1}{N(\varepsilon)} < \varepsilon,$$

where the first inequality follows by Exercise 1.2.7(b). Therefore, by the definition of limit of a sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty} \to 0$, as claimed. □

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Example 2.4

Example 2.4. Prove that $\{x_n\} = \{2 - 1/n^2\}$ has a limit of 2.

Proof. Let $\varepsilon > 0$. Define $N(\varepsilon) = 1/\sqrt{\varepsilon}$. Then for $n > N(\varepsilon) = 1/\sqrt{\varepsilon} > 0$, we have $1/n < \sqrt{\varepsilon}$ by Exercise 1.2.7(b), and so $1/n^2 < \varepsilon$ by Exercise 1.2.7(c) with $n = 2$. Therefore, for all $n > N(\varepsilon)$ we have

$$|x_n - L| = \left| \left(2 - \frac{1}{n^2} \right) - 2 \right| = \left| -\frac{1}{n^2} \right| = \frac{1}{n^2} < \varepsilon.$$

So, by the definition of the limit of a sequence, we have $L = 2$ and $\{x_n\} \to 2$, as claimed. □

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So, by the definition of the limit of a sequence, we have $L = 2$ and $\{x_n\} \to 2$, as claimed. □

Example 2.6

Example 2.6. Prove that $\{x_n\} = \{n^2\}$ diverges to ∞.

Proof. Let $M > 0$. Define $N(M) = \sqrt{M} + 1$. Then for all $n > N(M) = \sqrt{M} + 1$ we have by Exercise 1.2.7(c) (with $n = 2$)

$$x_n = n^2 > (\sqrt{M} + 1)^2 = M + 2\sqrt{M} + 1 > M.$$

Therefore by the definition of a sequence diverges to infinity, we have $\{x_n\} \to \infty$, as claimed. □

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Therefore by the definition of a sequence diverges to infinity, we have $\{x_n\} \to \infty$, as claimed. □

Theorem 2-1

Theorem 2-1. A sequence of real numbers can converge to at most one number.

Proof. We give a proof by contradiction. ASSUME that some sequence $\{x_n\}$ converges to both L and M, where $L \neq M$, say $L < M$. Let $\varepsilon = (M - L)/2$. Then the intervals $(L - \varepsilon, L + \varepsilon) = ((3L - M)/2, (L + M)/2)$ and $(M - \varepsilon, M + \varepsilon) = ((L + M)/2, (3M - L)/2)$ are disjoint. But since $\{x_n\} \to L$ then there is positive $N_L(\varepsilon) \in \mathbb{R}$ such that for all $n > N_L(\varepsilon)$ we have $|x_n - L| \leq \varepsilon$ (that is, $x_n \in (L - \varepsilon, L + \varepsilon)$), and since $\{x_n\} \to M$ then there is positive $N_M(\varepsilon) \in \mathbb{R}$ such that for all $n > N_M(\varepsilon)$ we have $|x_n - M| \leq \varepsilon$ (that is, $x_n \in (M - \varepsilon, M + \varepsilon)$).

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Theorem 2-2

Theorem 2-2. The sequence of real numbers $\{a_n\}$ converges to L if and only if for all $\varepsilon > 0$, all but a finite number of terms of $\{a_n\}$ lie in $(L - \varepsilon, L + \varepsilon)$.

Proof. First, suppose $\lim_{n\to\infty}\{a_n\} = L$ and let $\varepsilon > 0$ be given. Then, by the definition of limit of a sequence, there exists positive $N(\varepsilon) \in \mathbb{R}$ such that for all $n > N(\varepsilon)$ we have $|a_n - L| < \varepsilon$. That is, for all $n > N(\varepsilon)$ we have $a_n \in (L - \varepsilon, L + \varepsilon)$. Then all a_n lie in $(L - \varepsilon, L + \varepsilon)$, except possibly for $a_1, a_2, \ldots, a_{\lfloor N(\varepsilon)\rfloor}$. Hence, all but finitely many terms of $\{a_n\}$ lie in $(L - \varepsilon, L + \varepsilon)$, as claimed.

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Now suppose all but a finite number of terms of $\{a_n\}$ lie in $(L - \varepsilon, L + \varepsilon)$ where $\varepsilon > 0$ is given. Let $N(\varepsilon)$ be the largest subscript of such terms. Then for all $n > N(\varepsilon)$ we have $a_n \in (L - \varepsilon, L + \varepsilon)$. That is, for all $n > N(\varepsilon)$ we have $|a_n - L| < \varepsilon$. So by the definition of limit of a sequence, $\{a_n\} \to L$, as claimed. □

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Theorem 2-3

Theorem 2-3. If $\{a_n\}$ is a convergent sequence of real numbers, then the sequence $\{a_n\}$ is bounded.

Proof. Suppose $\{a_n\} \to L$. Then for $\varepsilon = 1$, there exists positive $N(\varepsilon) = N(1) = \in \mathbb{R}$ such that for all $n > N(\varepsilon)$ we have $|a_n - L| < \varepsilon = 1$ by the definition of limit of a sequence. Therefore $\max\{a_1, a_2, \ldots, a_{\lfloor N(1) \rfloor}, L + 1\}$ is an upper bound for $\{a_n\}$ and $\min\{a_1, a_2, \ldots, a_{\lfloor N(1) \rfloor}, L - 1\}$ is a lower bound for $\{a_n\}$. That is, sequence $\{a_n\}$ is bounded, as claimed. □

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Theorem 2-4

Theorem 2-4. Suppose $\{a_n\}$ and $\{b_n\}$ are sequences with $\{a_n\} \to a$ and $\{b_n\} \to b$. Then

(a) $\{a_n + b_n\} \to a + b$.

(b) $\{ca_n\} \to ca$ for any $c \in \mathbb{R}$.

(c) $\{a_n b_n\} \to ab$.

(d) If $b \neq 0$ and $b_n \neq 0$ for all $n \in \mathbb{N}$, then $\{a_n/b_n\} \to a/b$.

Proof. (a) This is our first "ε halves" proof. Let $\varepsilon > 0$ (notice then that $\varepsilon/2 > 0$ as well). Since $\{a_n\} \to a$ by hypothesis, then there is positive $N_a(\varepsilon/2) \in \mathbb{R}$ such that for all $n > N_a(\varepsilon/2)$ we have $|a_n - a| < \varepsilon/2$. Since $\{b_n\} \to b$ by hypothesis, then there is positive $N_b(\varepsilon/2) \in \mathbb{R}$ such that for all $n > N_b(\varepsilon/2)$ we have $|b_n - b| < \varepsilon/2$.

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Proof (continued).

$$\begin{aligned} |(a_n + b_n) - (a + b)| &= |(a_n - a) + (b_n - b)| \\ &\leq |a_n - a| + |b_n - b| \text{ by the Triangle Inequality} \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Therefore, by the definition of the limit of a sequence, we have $\{a_n + b_n\} \to a + b$, as claimed.

Theorem 2-4 (continued 2)

Theorem 2-4. Suppose $\{a_n\}$ and $\{b_n\}$ are sequences with $\{a_n\} \to a$ and $\{b_n\} \to b$. Then

(c) $\{a_n b_n\} \to ab$.

(d) If $b \neq 0$ and $b_n \neq 0$ for all $n \in \mathbb{N}$, then $\{a_n/b_n\} \to a/b$.

Proof (continued). (c) Let $\varepsilon > 0$. Since $\{a_n\}$ is convergent by hypothesis, the $\{a_n\}$ is bounded by Theorem 2-3, say $|a_n| < M$ for all $n \in \mathbb{N}$. Since $\{a_n\} \to a$ then, by the definition of the limit of a sequence, there is a positive $N_a(\varepsilon) \in \mathbb{R}$ such that if $n > N_a(\varepsilon)$ then we have $|a_n - a| < \dfrac{\varepsilon}{2|b|+1}$. (We'll discuss this choice for the bound in the notes. Also, we denote the parameter N for which we consider $n > N$ simply as $N_a(\varepsilon)$ instead of something more complicated as we did in part (a).) Since $\{b_n\} \to b$ by hypothesis, there is a positive $N_b(\varepsilon) \in \mathbb{R}$ such that if $n > N_b(\varepsilon)$ then we have $|b_n - b| < \dfrac{\varepsilon}{2M}$. (Again, we explain this choice in the notes.)

Theorem 2-4 (continued 2)

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Theorem 2-4 (continued 3)

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(d) If $b \neq 0$ and $b_n \neq 0$ for all $n \in \mathbb{N}$, then $\{a_n/b_n\} \to a/b$.

Proof (continued). (c) Let $N = \max\{N_a(\varepsilon), N_b(\varepsilon)\}$. Then for all $n > N$ we have that both $n > N_a(\varepsilon)$ and $n > N_b(\varepsilon)$ so that

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - a_n b + a_n b - ab| \\ &\leq |a_n b_n - a_n b| + |a_n b - ab| \text{ by the Triangle Inequality} \\ &= |a_n||b_n - b| + |b||a_n - a| \text{ by Theorem 1-13(d)} \\ &< M|b_n| + |b||a_n - a| \text{ since } \{a_n\} \text{ is bounded by } M \\ &< M\left(\frac{\varepsilon}{2M}\right) + |b|\frac{\varepsilon}{2|b|+1} \text{ since } n > N_a(\varepsilon) \text{ and } n > N_b(\varepsilon) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore $\{a_n b_n\} \to ab$ by the definition of limit of a sequence, as claimed.

Theorem 2-4 (continued 4)

Theorem 2-4. Suppose $\{a_n\}$ and $\{b_n\}$ are sequences with $\{a_n\} \to a$ and $\{b_n\} \to b$. Then

(d) If $b \neq 0$ and $b_n \neq 0$ for all $n \in \mathbb{N}$, then $\{a_n/b_n\} \to a/b$.

Proof (continued). (d) Let $\varepsilon > 0$. By Exercise 2.1.10, there is $M > 0$ such that $1/|b_n| < M$ for all $n \in \mathbb{N}$. Since $\{a_n\} \to a$ by hypothesis, there is positive $N_a(\varepsilon) \in \mathbb{R}$ such that if $n > N_a(\varepsilon)$ then $|a_n - a| < \dfrac{\varepsilon}{2M}$. Since $\{b_n\} \to b$ by hypothesis, there is positive $N_b(\varepsilon) \in \mathbb{R}$ such that if $n > N_b(\varepsilon)$ then $|b_n - b| < \dfrac{\varepsilon|a|}{2M|a|+1}$. Let $N = \max\{N_a(\varepsilon), N_b(\varepsilon)\}$.

$$\begin{aligned}
\left|\frac{a_n}{b_n} - \frac{a}{b}\right| &= \left|\frac{a_n b - a b_n}{b_n b}\right| = \left|\frac{a_n b - ab + ab - ab_n}{b_n b}\right| \\
&\leq \left|\frac{a_n b - ab}{b_n b}\right| + \left|\frac{ab - ab_n}{b_n b}\right| \text{ by the Triangle Inequality} \\
&= \frac{|b||a_n - a|}{|b_n b|} + \frac{|a||b_n - b|}{|b_n b|} \text{ by Theorem 1-13(d)}
\end{aligned}$$

Theorem 2-4 (continued 4)

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$$
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&\leq \left|\frac{a_n b - ab}{b_n b}\right| + \left|\frac{ab - ab_n}{b_n b}\right| \text{ by the Triangle Inequality} \\
&= \frac{|b||a_n - a|}{|b_n b|} + \frac{|a||b_n - b|}{|b_n b|} \text{ by Theorem 1-13(d)}
\end{aligned}
$$

Theorem 2-4 (continued 5)

Theorem 2-4. Suppose $\{a_n\}$ and $\{b_n\}$ are sequences with $\{a_n\} \to a$ and $\{b_n\} \to b$. Then

(d) If $b \neq 0$ and $b_n \neq 0$ for all $n \in \mathbb{N}$, then $\{a_n/b_n\} \to a/b$.

Proof (continued).

$$\begin{aligned} \left|\frac{a_n}{b_n} - \frac{a}{b}\right| &\leq \frac{|b||a_n - a|}{|b_n b|} + \frac{|a||b_n - b|}{|b_n b|} \text{ by Theorem 1-13(d)} \\ &= \frac{1}{|b_n|}|a_n - a| + \frac{|a|}{|b_n b|}|b_n - b| \\ &< M|a_n - a| + \frac{|a|M}{|b|}|b_n - b| \text{ since } \{1/b_n\} \text{ is bounded by } M \\ &< M\left(\frac{\varepsilon}{2M}\right) + \frac{|a|M}{|b|}\left(\frac{\varepsilon|b|}{2M|a|+1}\right) \text{ since } n > N_a(\varepsilon),\ n > N_b(\varepsilon) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore $\{a_n/b_n\} \to a/b$ by the definition of limit of a sequence. □

Theorem 2-6

Theorem 2-6. A bounded monotone sequence converges.

Proof. We give a proof for monotone increasing sequences, and leave the proof of monotone decreasing sequence for an exercise. Let $\{a_n\}$ be a monotone increasing, bounded sequence. Since the sequence forms a bounded set of real numbers, the it has a least upper bound (by the Axiom of Completeness), say L.

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Proof. We give a proof for monotone increasing sequences, and leave the proof of monotone decreasing sequence for an exercise. Let $\{a_n\}$ be a monotone increasing, bounded sequence. Since the sequence forms a bounded set of real numbers, the it has a least upper bound (by the Axiom of Completeness), say L. Then $a_n \leq L$ for all $n \in \mathbb{N}$ and by Theorem 1-15, for all $\varepsilon > 0$ there exists positive $N(\varepsilon) \in \mathbb{N}$ such that $a_{N(\varepsilon)} > L - \varepsilon$. Since $\{a_n\}$ is a monotone increasing sequence (bounded above by L), then for all $n > N(\varepsilon)$ we have $L \geq a_n \geq a_{N(\varepsilon)} > L - \varepsilon$ and there $|a_n - L| < \varepsilon$. That is, by the definition of the limit of a sequence, $\{a_n\} \to L$, as claimed. □

Theorem 2-6

Theorem 2-6. A bounded monotone sequence converges.

Proof. We give a proof for monotone increasing sequences, and leave the proof of monotone decreasing sequence for an exercise. Let $\{a_n\}$ be a monotone increasing, bounded sequence. Since the sequence forms a bounded set of real numbers, the it has a least upper bound (by the Axiom of Completeness), say L. Then $a_n \leq L$ for all $n \in \mathbb{N}$ and by Theorem 1-15, for all $\varepsilon > 0$ there exists positive $N(\varepsilon) \in \mathbb{N}$ such that $a_{N(\varepsilon)} > L - \varepsilon$. Since $\{a_n\}$ is a monotone increasing sequence (bounded above by L), then for all $n > N(\varepsilon)$ we have $L \geq a_n \geq a_{N(\varepsilon)} > L - \varepsilon$ and there $|a_n - L| < \varepsilon$. That is, by the definition of the limit of a sequence, $\{a_n\} \to L$, as claimed. □

Example 2.9

Example 2.9. Prove that the sequence $\{x_n\} = \{(1 + 1/n)^n\}$ is monotone increasing.

Proof. We apply the Binomial Theorem (Theorem 1-12). We have

$$\begin{aligned} x_n &= \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k (1)^{n-k} \\ &= \sum_{k=0}^{n} \frac{n(n-1)\cdots(n-k+1)}{k!} \left(\frac{1}{n}\right)^k \\ &= \sum_{k=0}^{n} \frac{1}{k!}(1)\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) \\ &\leq \sum_{k=0}^{n} \frac{1}{k!} \text{ since } \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) \leq 1. \end{aligned}$$

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Example 2.9 (continued)

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Proof (continued). Similarly,

$$x_{n+1} = \sum_{k=0}^{n+1} \frac{1}{k!}(1)\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right)\cdots\left(1 - \frac{k-1}{n+1}\right).$$

Now $1 - \dfrac{j}{n+1} > 1 - \dfrac{j}{n}$ for any $j > 0$. In the two sums, the kth term for n_{n+1} is greater than the kth term for x_n. In addition, x_{n+1} has one additional term (for $k = n+1$) so that $x_{n+1} > x_n$. So, by definition, $\{x_n\}$ is a monotone increasing sequence. □

Theorem 2-8

Theorem 2-8. Let A be a nonempty set of real numbers bounded above. Then there is a sequence $\{x_n\}$ such that (i) $x_n \in A$ for all $n \in \mathbb{N}$, and (ii) $\{x_n\} \to \text{lub}(A)$.

Proof. Since A is bounded above, the it has a least upper bound, say α (by the Axiom of Completeness). If $\alpha \in A$, let $x_n = \alpha$ for all $n \in \mathbb{N}$. If $\alpha \notin A$, then for all $n \in \mathbb{N}$ there exists $x_n \in A$ such that $\alpha - 1/n < x_n < \alpha$ by Theorem 1-16 (with $\varepsilon = 1/n$).

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$$\alpha - \varepsilon < \alpha - 1/N(\varepsilon) < \alpha - 1/n < x_n < \alpha.$$

That is, for all $n > N(\varepsilon)$ we have $|x_n - \alpha| < \varepsilon$. Then each $x_n \in A$ and $\{x_n\} \to \alpha = \text{lub}(A)$, by the definition of limit of a sequence. □

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