Analysis 1

Chapter 2. Sequences of Real Numbers

2-2. Subsequences—Proofs of Theorems

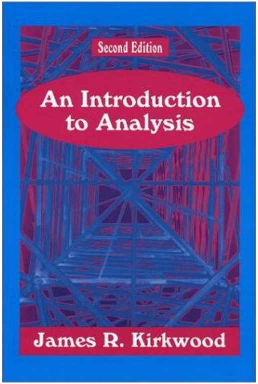

Table of contents

Theorem 2-10

Theorem 2-10. A sequence $\{a_n\}$ converges to L if and only if every subsequence of $\{a_n\}$ converges to L.

Proof. Suppose $\{a_n\}$ converges to L. Let $\varepsilon > 0$. Then there exists positive $N(\varepsilon) \in \mathbb{R}$ such that for all $n > N(\varepsilon)$, we have $|a_n - L| < \varepsilon$. Let $\{a_{n_k}\}$ be a subsequence of $\{a_n\}$, and let $K(\varepsilon) = N(\varepsilon)$. Then for all $k \geq K(\varepsilon)$ we have $n_k \geq k \geq K(\varepsilon) = N(\varepsilon)$ ($n_k \geq k$, as observed in Note 2.2.A) and hence $|a_{n_k} - L| < \varepsilon$ (since $n_k \geq N(\varepsilon)$). That is, $\{a_{n_k}\}$ converges to L. Since $\{a_{n_k}\}$ is an arbitrary subsequence of $\{a_n\}$, then every subsequence of $\{a_n\}$ converges to L.

Theorem 2-10

Theorem 2-10. A sequence $\{a_n\}$ converges to L if and only if every subsequence of $\{a_n\}$ converges to L.

Proof. Suppose $\{a_n\}$ converges to L. Let $\varepsilon > 0$. Then there exists positive $N(\varepsilon) \in \mathbb{R}$ such that for all $n > N(\varepsilon)$, we have $|a_n - L| < \varepsilon$. Let $\{a_{n_k}\}$ be a subsequence of $\{a_n\}$, and let $K(\varepsilon) = N(\varepsilon)$. Then for all $k \geq K(\varepsilon)$ we have $n_k \geq k \geq K(\varepsilon) = N(\varepsilon)$ ($n_k \geq k$, as observed in Note 2.2.A) and hence $|a_{n_k} - L| < \varepsilon$ (since $n_k \geq N(\varepsilon)$). That is, $\{a_{n_k}\}$ converges to L. Since $\{a_{n_k}\}$ is an arbitrary subsequence of $\{a_n\}$, then every subsequence of $\{a_n\}$ converges to L.

Suppose every subsequence of $\{a_n\}$ converges to L. With $n_k = k$ we have $\{a_{n_k}\} = \{a_k\}_{k=1}^{\infty} = \{a_n\}_{n=1}^{\infty}$; that is, $\{a_n\}$ is a subsequence of itself. Therefore $\{a_n\}$ converges to L. $\square$

Theorem 2-10

Theorem 2-10. A sequence $\{a_n\}$ converges to L if and only if every subsequence of $\{a_n\}$ converges to L.

Proof. Suppose $\{a_n\}$ converges to L. Let $\varepsilon > 0$. Then there exists positive $N(\varepsilon) \in \mathbb{R}$ such that for all $n > N(\varepsilon)$, we have $|a_n - L| < \varepsilon$. Let $\{a_{n_k}\}$ be a subsequence of $\{a_n\}$, and let $K(\varepsilon) = N(\varepsilon)$. Then for all $k \geq K(\varepsilon)$ we have $n_k \geq k \geq K(\varepsilon) = N(\varepsilon)$ ($n_k \geq k$, as observed in Note 2.2.A) and hence $|a_{n_k} - L| < \varepsilon$ (since $n_k \geq N(\varepsilon)$). That is, $\{a_{n_k}\}$ converges to L. Since $\{a_{n_k}\}$ is an arbitrary subsequence of $\{a_n\}$, then every subsequence of $\{a_n\}$ converges to L.

Suppose every subsequence of $\{a_n\}$ converges to L. With $n_k = k$ we have $\{a_{n_k}\} = \{a_k\}_{k=1}^{\infty} = \{a_n\}_{n=1}^{\infty}$; that is, $\{a_n\}$ is a subsequence of itself. Therefore $\{a_n\}$ converges to L. $\square$

Theorem 2-11

Theorem 2-11. Real number L is a subsequential limit of $\{a_n\}$ if and only if $\varepsilon > 0$, the interval $(L-\varepsilon, L+\varepsilon)$ contains infinitely many terms of $\{a_n\}$.

Proof. Suppose L is a subsequential limit of $\{a_n\}$. Then there is a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ that converges to L. Let $\varepsilon > 0$. Then by the definition of limit of a subsequence, there is positive $K(\varepsilon) \in \mathbb{R}$ such that if $k > K(\varepsilon)$, then $|a_{n_k} - L| < \varepsilon$. That is, $a_{n_k} \in (L-\varepsilon, L+\varepsilon)$ for all $k > K(\varepsilon)$. Therefore, $(L-\varepsilon, L+\varepsilon)$ contains infinitely many terms of $\{a_n\}$, namely $a_{n_{\lceil K(\varepsilon)\rceil}}, a_{n_{\lceil K(\varepsilon)\rceil+1}}, a_{n_{\lceil K(\varepsilon)\rceil+2}}, \ldots$, as claimed.

Theorem 2-11

Theorem 2-11. Real number L is a subsequential limit of $\{a_n\}$ if and only if $\varepsilon > 0$, the interval $(L-\varepsilon, L+\varepsilon)$ contains infinitely many terms of $\{a_n\}$.

Proof. Suppose L is a subsequential limit of $\{a_n\}$. Then there is a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ that converges to L. Let $\varepsilon > 0$. Then by the definition of limit of a subsequence, there is positive $K(\varepsilon) \in \mathbb{R}$ such that if $k > K(\varepsilon)$, then $|a_{n_k} - L| < \varepsilon$. That is, $a_{n_k} \in (L-\varepsilon, L+\varepsilon)$ for all $k > K(\varepsilon)$. Therefore, $(L-\varepsilon, L+\varepsilon)$ contains infinitely many terms of $\{a_n\}$, namely $a_{n_{\lceil K(\varepsilon)\rceil}}, a_{n_{\lceil K(\varepsilon)\rceil+1}}, a_{n_{\lceil K(\varepsilon)\rceil+2}}, \ldots$, as claimed.

Conversely, suppose for every $\varepsilon > 0$ that the interval $(L-\varepsilon, L+\varepsilon)$ contains infinitely many terms of $\{a_n\}$. We inductively construct a subsequence of $\{a_n\}$ that converges to L. For $k = 1$ we take $\varepsilon = 1$ and, since there are infinitely many terms of $\{a_n\}$ in $(L-1, L+1)$, we choose one and denote it a_{n_1}. For $k = 2$ we take $\varepsilon = 1/2$ and, since there are infinitely many terms of $\{a_n\}$ in $(L-1/2, L+1/2)$, we choose one with subscript greater than n_1 and denote it a_{n_2}.

Theorem 2-11

Theorem 2-11. Real number L is a subsequential limit of $\{a_n\}$ if and only if $\varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ contains infinitely many terms of $\{a_n\}$.

Proof. Suppose L is a subsequential limit of $\{a_n\}$. Then there is a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ that converges to L. Let $\varepsilon > 0$. Then by the definition of limit of a subsequence, there is positive $K(\varepsilon) \in \mathbb{R}$ such that if $k > K(\varepsilon)$, then $|a_{n_k} - L| < \varepsilon$. That is, $a_{n_k} \in (L - \varepsilon, L + \varepsilon)$ for all $k > K(\varepsilon)$. Therefore, $(L - \varepsilon, L + \varepsilon)$ contains infinitely many terms of $\{a_n\}$, namely $a_{n_{\lceil K(\varepsilon)\rceil}}, a_{n_{\lceil K(\varepsilon)\rceil+1}}, a_{n_{\lceil K(\varepsilon)\rceil+2}}, \ldots$, as claimed.

Conversely, suppose for every $\varepsilon > 0$ that the interval $(L - \varepsilon, L + \varepsilon)$ contains infinitely many terms of $\{a_n\}$. We inductively construct a subsequence of $\{a_n\}$ that converges to L. For $k = 1$ we take $\varepsilon = 1$ and, since there are infinitely many terms of $\{a_n\}$ in $(L - 1, L + 1)$, we choose one and denote it a_{n_1}. For $k = 2$ we take $\varepsilon = 1/2$ and, since there are infinitely many terms of $\{a_n\}$ in $(L - 1/2, L + 1/2)$, we choose one with subscript greater than n_1 and denote it a_{n_2}.

Theorem 2-11 (continued)

Theorem 2-11. Real number L is a subsequential limit of $\{a_n\}$ if and only if $\varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ contains infinitely many terms of $\{a_n\}$.

Proof (continued). We take these as the Base Cases. Now suppose we have similarly chosen $a_{n_1}, a_{n_2}, \ldots, a_{n_k}$ where $n_1 < n_2 < \cdots < n_k$ and each $a_{n_i} \in (L - 1/i, L + 1/i)$ for $1 \leq i \leq k$ (this is the Induction Hypothesis). For the Induction Step, consider $i = k + 1$ and let $\varepsilon = 1/(k+1)$. Since there are infinitely many terms of $\{a_n\}$ in $(L - 1/(k+1), L + 1/(k+1))$, we can choose one with subscript greater than n_k (this is the Induction Step). We have produced a subsequence $\{a_{n_k}\}$ of $\{a_n\}$. We still need to show that $\{a_{n_k}\}$ converges to L.

Let $\varepsilon > 0$. By the Archimedean Principle (Theorem 1-18), there is $K(\varepsilon) \in \mathbb{N}$ such that $1/K(\varepsilon) < \varepsilon$. If $k > K(\varepsilon)$ then $1/k < \varepsilon$ and $a_{n_k} \in (L - 1/k, L + 1/k) \subset (L - \varepsilon, L + \varepsilon)$. Therefore, by the definition of limit of a subsequence we have $\{a_{n_k}\}$ converges to L. That is, L is a subsequential limit of $\{a_n\}$, as claimed. □

Theorem 2-11 (continued)

Theorem 2-11. Real number L is a subsequential limit of $\{a_n\}$ if and only if $\varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ contains infinitely many terms of $\{a_n\}$.

Proof (continued). We take these as the Base Cases. Now suppose we have similarly chosen $a_{n_1}, a_{n_2}, \ldots, a_{n_k}$ where $n_1 < n_2 < \cdots < n_k$ and each $a_{n_i} \in (L - 1/i, L + 1/i)$ for $1 \leq i \leq k$ (this is the Induction Hypothesis). For the Induction Step, consider $i = k + 1$ and let $\varepsilon = 1/(k + 1)$. Since there are infinitely many terms of $\{a_n\}$ in $(L - 1/(k + 1), L + 1/(k + 1))$, we can choose one with subscript greater than n_k (this is the Induction Step). We have produced a subsequence $\{a_{n_k}\}$ of $\{a_n\}$. We still need to show that $\{a_{n_k}\}$ converges to L.

Let $\varepsilon > 0$. By the Archimedean Principle (Theorem 1-18), there is $K(\varepsilon) \in \mathbb{N}$ such that $1/K(\varepsilon) < \varepsilon$. If $k > K(\varepsilon)$ then $1/k < \varepsilon$ and $a_{n_k} \in (L - 1/k, L + 1/k) \subset (L - \varepsilon, L + \varepsilon)$. Therefore, by the definition of limit of a subsequence we have $\{a_{n_k}\}$ converges to L. That is, L is a subsequential limit of $\{a_n\}$, as claimed. □

Exercise 2.2.8(a)

Exercise 2.2.8(a). Construct a sequence with exactly two subsequential limits. Can this be done is such a way that no two terms of the sequence are the same?

Solution. Consider the sequence $\{a_n\} = \{(-1)^n\}$. Since for $1 > \varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ can contain at most one of -1 and 1 (and these are the only terms of the sequence), then these are the only possible subsequential limits. The subsequence $\{a_{2n-1}\}_{n=1}^{\infty} = \{-1, -1, -1, \ldots\}$ converges to -1, and the subsequence $\{a_{2n}\}_{n=1}^{\infty} = \{1, 1, 1, \ldots\}$ converges to 1. So there are exactly two subsequential limits of $\{a_n\}$.

Exercise 2.2.8(a)

Exercise 2.2.8(a). Construct a sequence with exactly two subsequential limits. Can this be done is such a way that no two terms of the sequence are the same?

Solution. Consider the sequence $\{a_n\} = \{(-1)^n\}$. Since for $1 > \varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ can contain at most one of -1 and 1 (and these are the only terms of the sequence), then these are the only possible subsequential limits. The subsequence $\{a_{2n-1}\}_{n=1}^{\infty} = \{-1, -1, -1, \ldots\}$ converges to -1, and the subsequence $\{a_{2n}\}_{n=1}^{\infty} = \{1, 1, 1, \ldots\}$ converges to 1. So there are exactly two subsequential limits of $\{a_n\}$.

This can be done in such a way that no two terms are the same. Let

$$f(n) = \begin{cases} -1 - 1/n & n \text{ odd} \\ 1 + 1/n & n \text{ even} \end{cases}$$

and define $\{a_n\} = \{f(n)\}$. Then no two terms of $\{a_n\}$ are equal.

Exercise 2.2.8(a)

Exercise 2.2.8(a). Construct a sequence with exactly two subsequential limits. Can this be done is such a way that no two terms of the sequence are the same?

Solution. Consider the sequence $\{a_n\} = \{(-1)^n\}$. Since for $1 > \varepsilon > 0$, the interval $(L-\varepsilon, L+\varepsilon)$ can contain at most one of -1 and 1 (and these are the only terms of the sequence), then these are the only possible subsequential limits. The subsequence $\{a_{2n-1}\}_{n=1}^{\infty} = \{-1, -1, -1, \ldots\}$ converges to -1, and the subsequence $\{a_{2n}\}_{n=1}^{\infty} = \{1, 1, 1, \ldots\}$ converges to 1. So there are exactly two subsequential limits of $\{a_n\}$.

This can be done in such a way that no two terms are the same. Let

$$f(n) = \begin{cases} -1 - 1/n & n \text{ odd} \\ 1 + 1/n & n \text{ even} \end{cases}$$

and define $\{a_n\} = \{f(n)\}$. Then no two terms of $\{a_n\}$ are equal.

Exercise 2.2.8(a) (continued)

Exercise 2.2.8(a). Construct a sequence with exactly two subsequential limits. Can this be done is such a way that no two terms of the sequence are the same?

Solution (continued). The only intervals of the form $(L-\varepsilon, L+\varepsilon)$ which contain infinitely many terms of $\{a_n\}$ for all $\varepsilon > 0$ are the ones for which $L = -1$ or $L = 1$, so by Theorem 2-11 the only possible subsequential limits are -1 and 1. Let $\varepsilon > 0$. Define $N_o = N_e = 1/\varepsilon$. Then for all $n \geq N_o$ we have $1/n < \varepsilon$ and so the interval $(-1-\varepsilon, -1+\varepsilon)$ contains infinitely many terms of $\{a_n\}$, namely those of odd subscripts where the subscripts are greater than N_o. Similarly for all $n \geq N_e$ we have $1/n < \varepsilon$ and so the interval $(1-\varepsilon, 1+\varepsilon)$ contains infinitely many terms of $\{a_n\}$, namely those of even subscripts where the subscripts are greater than N_e. $\square$

Exercise 2.2.8(a) (continued)

Exercise 2.2.8(a). Construct a sequence with exactly two subsequential limits. Can this be done is such a way that no two terms of the sequence are the same?

Solution (continued). The only intervals of the form $(L-\varepsilon, L+\varepsilon)$ which contain infinitely many terms of $\{a_n\}$ for all $\varepsilon > 0$ are the ones for which $L = -1$ or $L = 1$, so by Theorem 2-11 the only possible subsequential limits are -1 and 1. Let $\varepsilon > 0$. Define $N_o = N_e = 1/\varepsilon$. Then for all $n \geq N_o$ we have $1/n < \varepsilon$ and so the interval $(-1-\varepsilon, -1+\varepsilon)$ contains infinitely many terms of $\{a_n\}$, namely those of odd subscripts where the subscripts are greater than N_o. Similarly for all $n \geq N_e$ we have $1/n < \varepsilon$ and so the interval $(1-\varepsilon, 1+\varepsilon)$ contains infinitely many terms of $\{a_n\}$, namely those of even subscripts where the subscripts are greater than N_e.
□

Exercise 2.2.12(a)

Exercise 2.2.12(a). If $\{a_n\} \to L$ and if $a_n \leq L$ for infinitely many values of n, then there is a subsequence of $\{a_n\}$ that is increasing (i.e., nondecreasing) and converges to L.

Proof. Let $\{a_n\} \to L$ and suppose $a_n \leq L$ for infinitely many values of n. If $a_n = L$ for infinitely many values of n, then we can simply take a constant subsequence of all L's to get the desired subsequence. So without loss of generality ("WLOG"), we can assume that only finitely many $a_n = L$; say the last one has subscript N (take $N = 0$ if none of the a_n equal L).

Exercise 2.2.12(a)

Exercise 2.2.12(a). If $\{a_n\} \to L$ and if $a_n \leq L$ for infinitely many values of n, then there is a subsequence of $\{a_n\}$ that is increasing (i.e., nondecreasing) and converges to L.

Proof. Let $\{a_n\} \to L$ and suppose $a_n \leq L$ for infinitely many values of n. If $a_n = L$ for infinitely many values of n, then we can simply take a constant subsequence of all L's to get the desired subsequence. So without loss of generality ("WLOG"), we can assume that only finitely many $a_n = L$; say the last one has subscript N (take $N = 0$ if none of the a_n equal L). Similar to the proof of Theorem 2-11 we inductively construct an increasing subsequence $\{a_{n_k}\}$ of $\{a_n\}$ that converges to L. For $k = 1$, since there are infinitely many $a_n \leq L$, we can choose one with subscript greater than N and denote it a_{n_1}. For $k = 2$, since there are infinitely many $a_n \leq L$ and $\{a_n\} \to L$, we can choose an a_n with subscript greater than n_1, satisfying $|a_n - L| < \min\{1/2, L - a_{n_1}\}$, and denote it a_{n_2}. We take these as the Base Cases.

Exercise 2.2.12(a)

Exercise 2.2.12(a). If $\{a_n\} \to L$ and if $a_n \leq L$ for infinitely many values of n, then there is a subsequence of $\{a_n\}$ that is increasing (i.e., nondecreasing) and converges to L.

Proof. Let $\{a_n\} \to L$ and suppose $a_n \leq L$ for infinitely many values of n. If $a_n = L$ for infinitely many values of n, then we can simply take a constant subsequence of all L's to get the desired subsequence. So without loss of generality ("WLOG"), we can assume that only finitely many $a_n = L$; say the last one has subscript N (take $N = 0$ if none of the a_n equal L). Similar to the proof of Theorem 2-11 we inductively construct an increasing subsequence $\{a_{n_k}\}$ of $\{a_n\}$ that converges to L. For $k = 1$, since there are infinitely many $a_n \leq L$, we can choose one with subscript greater than N and denote it a_{n_1}. For $k = 2$, since there are infinitely many $a_n \leq L$ and $\{a_n\} \to L$, we can choose an a_n with subscript greater than n_1, satisfying $|a_n - L| < \min\{1/2, L - a_{n_1}\}$, and denote it a_{n_2}. We take these as the Base Cases.

Exercise 2.2.12(a) (continued)

Proof (continued). Now suppose we have similarly chosen $a_{n_1}, a_{n_2}, \ldots, a_{n_k}$ where $n_1 < n_2 < \cdots < n_k$ and $|a_{n_i} - L| < \min\{1/i, L - a_{n_{i-1}}\}$ for $2 \le i \le k$ (this is the Induction Hypothesis). For the Induction Step, consider $i = k + 1$. Since there are infinitely many $a_n \le L$ and $\{a_n\} \to L$, we can choose an a_n with subscript greater than n_k, satisfying $|a_n - L| < \min\{1/(k+1), L - a_{n_k}\}$, and denote it $a_{n_{k+1}}$ (this is the Induction Step). We have produced an increasing subsequence $\{a_{n_k}\}$ of $\{a_n\}$. We still need to show that $\{a_{n_k}\}$ converges to L.

Let $\varepsilon > 0$. By the Archimedean Principle (Theorem 1-18), there is $K(\varepsilon) \in \mathbb{N}$ such that $1/K(\varepsilon) < \varepsilon$. If $k > K(\varepsilon)$ then $1/k < \varepsilon$ and $a_{n_k} \in (L - 1/k, L + 1/k) \subset (L - \varepsilon, L + \varepsilon)$. Hence, by the definition of limit of a subsequence we have $\{a_{n_k}\}$ converges to L. Therefore, we have constructed increasing subsequence $\{a_{n_k}\}$ of $\{a_n\}$ that converges to L, as claimed. □

Exercise 2.2.12(a) (continued)

Proof (continued). Now suppose we have similarly chosen $a_{n_1}, a_{n_2}, \ldots, a_{n_k}$ where $n_1 < n_2 < \cdots < n_k$ and $|a_{n_i} - L| < \min\{1/i, L - a_{n_{i-1}}\}$ for $2 \leq i \leq k$ (this is the Induction Hypothesis). For the Induction Step, consider $i = k + 1$. Since there are infinitely many $a_n \leq L$ and $\{a_n\} \to L$, we can choose an a_n with subscript greater than n_k, satisfying $|a_n - L| < \min\{1/(k+1), L - a_{n_k}\}$, and denote it $a_{n_{k+1}}$ (this is the Induction Step). We have produced an increasing subsequence $\{a_{n_k}\}$ of $\{a_n\}$. We still need to show that $\{a_{n_k}\}$ converges to L.

Let $\varepsilon > 0$. By the Archimedean Principle (Theorem 1-18), there is $K(\varepsilon) \in \mathbb{N}$ such that $1/K(\varepsilon) < \varepsilon$. If $k > K(\varepsilon)$ then $1/k < \varepsilon$ and $a_{n_k} \in (L - 1/k, L + 1/k) \subset (L - \varepsilon, L + \varepsilon)$. Hence, by the definition of limit of a subsequence we have $\{a_{n_k}\}$ converges to L. Therefore, we have constructed increasing subsequence $\{a_{n_k}\}$ of $\{a_n\}$ that converges to L, as claimed. $\square$