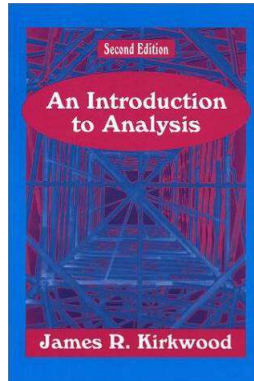


Analysis 1

Chapter 2. Sequences of Real Numbers

2-3. The Bolzano-Weierstrass Theorem—Proofs of Theorems



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Theorem 2-12

Theorem 2-12. Bolzano-Weierstrass Theorem.

Every bounded infinite set of real numbers has at least one limit point.

Proof. Let A be a bounded set of real numbers. Since A is bounded, then there is a positive $M \in \mathbb{R}$ such that $A \subset [-M, M] = A_0$. Then A_0 contains an infinite number of points in A . Cut A_0 into two closed subintervals of equal length, $[-M, 0]$ and $[0, M]$. Since A_0 contains an infinite number of points in A , the either $[-M, 0]$ or $[0, M]$ contains an infinite number of points in A ; denote the set containing an infinite number of points in A as A_1 and notice that the length of A_1 is M . Divide A_1 into two closed subintervals of equal length, and choose one of these subintervals that contains infinitely many points of A and denote it as A_2 ; notice the length of A_2 is $M/2$. Inductively create the sequence of sets A_k for each $k \in \mathbb{N}$ such that A_k contains infinitely many points of A and the length of A_k is $M/2^{k-1}$.

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Theorem 2-12 (continued 1)

Theorem 2-12. Bolzano-Weierstrass Theorem.

Every bounded infinite set of real numbers has at least one limit point.

Proof (continued). Each A_k is a closed interval, so denote $A_k = [a_k, b_k]$. By construction, $A_k \supset A_{k+1}$ and $\lim_{k \rightarrow \infty} (b_k - a_k) = \lim_{k \rightarrow \infty} M/2^{k-1} = 0$ by Exercise 2.2.2(c). Then by Theorem 2-7, $\bigcap_{n=1}^{\infty} A_n = \{p\}$ for some p . Next, we prove that p is a limit point of set A .

Let $\varepsilon > 0$. Choose $N(\varepsilon) = N \in \mathbb{N}$ such that $M/2^{N-1} < \varepsilon$ (such $N(\varepsilon)$ exists by the definition of the limit of a sequence, since $\lim_{k \rightarrow \infty} M/2^{k-1} = 0$). Now $p \in A_N$, since p is in A_k for all $k \in \mathbb{N}$, and the length of A_N is $M/2^{N-1}$. Therefore,

$$(p - \varepsilon, p + \varepsilon) \supset \left[p - \frac{M}{2^{N-1}}, p + \frac{M}{2^{N-1}} \right] \supset A_N.$$

This is illustrated below.

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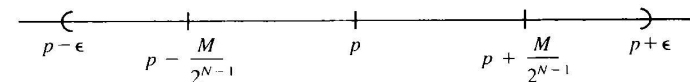
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Theorem 2-12 (continued 2)

Theorem 2-12. Bolzano-Weierstrass Theorem.

Every bounded infinite set of real numbers has at least one limit point.

Proof (continued).



Kirkwood's Figure 2-6

Hence, $(p - \varepsilon, p + \varepsilon)$ contains infinitely many points of A (since A_N does). Since $\varepsilon > 0$ is arbitrary, then by the definition of limit point of a set, we have that p is a limit point of set A . That is, set A has at least one limit point, as claimed. \square

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Theorem 2-14

Theorem 2-14. Every bounded sequence has a convergent subsequence.

Proof. Let $\{a_n\}$ be a bounded sequence. First, suppose there is a number L for which an infinite number of terms of $\{a_n\}$ equal L , say $a_{n_1} = a_{n_2} = a_{n_3} = \dots = L$ where $n_1 < n_2 < n_3 < \dots$. Then the constant subsequence $\{a_{n_k}\} = \{a_{n_1}, a_{n_2}, a_{n_3}, \dots\}$ converges to L .

If there is no such L repeated an infinite number of times in $\{a_n\}$, then $\{a_n\}$ but contain an infinite number of different terms. Let set A be the set of terms in $\{a_n\}$. Then A is an infinite set and, by hypothesis, it is bounded. By the Bolzano-Weierstrass Theorem (Theorem 2-12), there is a limit point p of set A . So by the definition of limit point of a set, for all $\varepsilon > 0$ the interval $(p - \varepsilon, p + \varepsilon)$ contains an infinite number of points in set A . That is, $(p - \varepsilon, p + \varepsilon)$ contains an infinite number of terms of $\{a_n\}$. Then by Theorem 2-11, p is a subsequential limit of $\{a_n\}$; that is, bounded sequence $\{a_n\}$ has a convergent subsequence as claimed. \square

Theorem 2-15 (continued)

Theorem 2-15.

- (a) A sequence that is unbounded above has a subsequence that diverges to $+\infty$.
- (b) A sequence that is unbounded below has a subsequence that diverges to $-\infty$.

Proof (continued). For the Induction Step, we consider $n = \ell + 1$. Notice that there are infinitely many terms of $\{a_n\}$ greater than $\ell + 1$, for if there were not then we could choose the last such term greater than $\ell + 1$, say a_m , and then the sequence would be bounded above by $\max\{a_1, a_2, \dots, a_m, \ell + 1\}$, contradicting the hypothesized unbounded above property of $\{a_n\}$. So there is some term $a_{n_{\ell+1}}$ in $\{a_n\}$ where $n_{\ell+1} > n_\ell$ and $a_{n_{\ell+1}} > \ell + 1$. That is, $P(k)$ holds for $k = \ell + 1$ and the Induction Step is established. So, by the Induction Principle, $P(k)$ holds for all $k \in \mathbb{N}$. Therefore, there is a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $a_{n_k} > k$ for each $k \in \mathbb{N}$. By the definition of diverge to infinity, subsequence $\{a_{n_k}\}$ diverges to ∞ , as claimed.

Theorem 2-15

Theorem 2-15.

- (a) A sequence that is unbounded above has a subsequence that diverges to $+\infty$.
- (b) A sequence that is unbounded below has a subsequence that diverges to $-\infty$.

Proof. (a) Let $\{a_n\}$ be sequence that is unbounded above. We give an inductive construction of a subsequence of $\{a_n\}$ that diverges to ∞ , as Kirkwood does (we could also give a recursive construction of the subsequence).

Let $P(k)$ be the statement that there is a positive integer $n_k > n_{k-1}$ such that $a_{n_k} > k$. For the Base Case observe that, since $\{a_n\}$ is unbounded above, there is a term greater than 1; denote such a term as a_{n_1} . Then $a_{n_1} > 1$ and $P(1)$ is true. For the Induction Hypothesis, suppose that $P(k)$ holds for all $k \in \mathbb{N}$ with $k \leq \ell$.

Exercise 2.3.16(a)

Exercise 2.3.16(a). Let $\{a_n\}$ be a sequence. Then $\limsup a_n = \overline{\lim} a_n$ is a subsequential limit of $\{a_n\}$.

Proof. First, suppose $\overline{\lim} a_n = +\infty$. Let P be the set of (real number) subsequential limits of $\{a_n\}$. Then P is not bounded above (or else, by the Axiom of Completeness, P would have a real number lub). For each $k \in \mathbb{N}$, there must be infinitely many terms of $\{a_n\}$ greater than $k + 1$, for if there were not then we could choose the last such term greater than k , say a_m , and then the sequence would be bounded above by $\max\{a_1, a_2, \dots, a_m, k + 1\}$, contradicting fact that P is not bounded above. For $k = 1$, there is some term of $\{a_n\}$ greater than 1; denote it as a_{n_1} . Recursively define $\{a_{n_k}\}$ by letting $a_{n_{k+1}}$ be greater than $k + 1$ and $n_{k+1} > n_k$ (such $a_{n_{k+1}}$ exists since there are infinitely many terms of $\{a_n\}$ greater than $k + 1$). Then for any $M > 0$, there is $k \in \mathbb{N}$ with $k \geq M$ (by the Archimedean Principle, Theorem 1-18), so that $\{a_{n_k}\} \rightarrow +\infty$ (by the definition of a sequence diverges to $+\infty$). That is, $\overline{\lim} a_n = +\infty$ is a subsequential limit of $\{a_n\}$, as claimed.

Exercise 2.3.16(a) (continued 1)

Exercise 2.3.16(a). Let $\{a_n\}$ be a sequence. Then $\limsup a_n = \overline{\lim} a_n$ is a subsequential limit of $\{a_n\}$.

Proof (continued). Second, suppose $\overline{\lim} a_n = -\infty$. Let P be the set of (real number) subsequential limits of $\{a_n\}$. Set P must be bounded above, or else $\overline{\lim} a_n = +\infty$ by the argument given above. If P is bounded below then P has a real number glb by the Axiom of Completeness and, since $\text{glb}(P) \leq \text{lub}(P)$, we cannot have $\text{lub}(P) = \overline{\lim} a_n = -\infty$. Hence P cannot be bounded below and for an real number $-K - 1$ there are infinitely many terms of $\{a_n\}$ less than $-K - 1$. For $k = 1$, there is some term of $\{a_n\}$ less than -1 ; denote it as a_{n_1} . Recursively define $\{a_{n_k}\}$ by letting $a_{n_{k+1}}$ be less than $-k - 1$ and $n_{k+1} > n_k$ (such $a_{n_{k+1}}$ exists since there are infinitely many terms of $\{a_n\}$ less than $-k - 1$). Then for any number K , there is $k \in \mathbb{N}$ with $-k \leq K$ (by the Archimedean Principle and Theorem 1-7(d)), so that $\{a_{n_k}\} \rightarrow -\infty$ (by the definition of a sequence diverges to $-\infty$). That is, $\overline{\lim} a_n = -\infty$ is a subsequential limit of $\{a_n\}$, as claimed.

Exercise 2.3.16(a) (continued 2)

Exercise 2.3.16(a). Let $\{a_n\}$ be a sequence. Then $\limsup a_n = \overline{\lim} a_n$ is a subsequential limit of $\{a_n\}$.

Proof (continued). Suppose $\overline{\lim} a_n = b \in \mathbb{R}$. Let P be the set of (real number) subsequential limits of $\{a_n\}$. We will construct a subsequence which converges to b . Since (by definition of \limsup) $b = \text{lub } P$ then for $\varepsilon = 1$ there exists $b_1 \in P$ such that $|b - b_1| < 1/2$ by Theorem 1-15(a). Since b_1 is a subsequential limit of $\{a_n\}$, then there is term a_{n_1} of $\{a_n\}$ such that $|b_1 - a_{n_1}| < 1/2$ by Theorem 2-11. Then

$$|b - a_{n_1}| = |b - b_1 + b_1 - a_{n_1}| \leq |b - b_1| + |b_1 - a_{n_1}| < 1/2 + 1/2 = 1.$$

We now create subsequence a_{n_k} recursively. Let $k \in \mathbb{N}$ with $k > 1$. Consider $\varepsilon = 1/k$ and choose $b_k \in P$ such that $|b - b_k| < 1/(2k)$ (which can be done by Theorem 1-15(a)). There is a term a_{n_k} of $\{a_n\}$ such that $|b_k - a_{n_k}| < 1/(2k)$ and $n_k > n_{k-1}$ (which can be done by Corollary 2-11).

Exercise 2.3.16(a) (continued 3)

Exercise 2.3.16(a). Let $\{a_n\}$ be a sequence. Then $\limsup a_n = \overline{\lim} a_n$ is a subsequential limit of $\{a_n\}$.

Proof (continued). Then

$$|b - a_{n_k}| = |b - b_k + b_k - a_{n_k}| \leq |b - b_k| + |b_k - a_{n_k}| < 1/(2k) + 1/(2k) = 1/k.$$

For any $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that $1/k < \varepsilon$ by the Archimedean Principle (Theorem 1-18). By the definition of limit of a sequence $\{a_{n_k}\} \rightarrow b$, so that b is a subsequential limit of $\{a_n\}$. That is, $b = \overline{\lim} a_n$ is a subsequential limit of $\{a_n\}$, as claimed. \square

Theorem 2-17(a)

Theorem 2-17. Let $\{a_n\}$ be a bounded sequence. Then

- (a) $\overline{\lim} a_n = L$ if and only if for all $\varepsilon > 0$, there exists infinitely many terms of $\{a_n\}$ in $(L - \varepsilon, L + \varepsilon)$ but only finitely many terms of $\{a_n\}$ with $a_n > L + \varepsilon$.

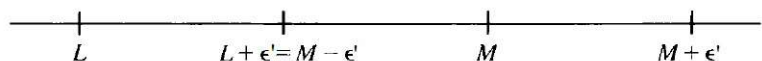
Proof. (a) First, let $\overline{\lim} a_n = L$ and let $\varepsilon > 0$ be given. Since L is a subsequential limit of $\{a_n\}$ by Exercise 2.3.16(a), then by Theorem 2-11 there are infinitely many terms of $\{a_n\}$ in $(L - \varepsilon, L + \varepsilon)$. ASSUME there are an infinite number of terms of $\{a_n\}$ with $a_n > L + \varepsilon$. Since $\{a_n\}$ is bounded above, say by M , there are infinitely many terms of $\{a_n\}$ between $L + \varepsilon$ and M . By Exercise 2.3.5, there is a subsequence $\{a_{n_k}\}$ with a limit at least as large as $L + \varepsilon$. But this means that $L + \varepsilon$ is a subsequential limit greater than $\overline{\lim} a_n = L$, a CONTRADICTION since L is an upper bound for the set of subsequential limits. This contradiction shows that the assumption that there are an infinite number of terms of $\{a_n\}$ with $a_n > L + \varepsilon$ is false. Hence, there are only a finite number of terms of $\{a_n\}$ with $a_n > L + \varepsilon$, as claimed.

Theorem 2-17(a) (continued)

Theorem 2-17. Let $\{a_n\}$ be a bounded sequence. Then

- (a) $\lim a_n = L$ if and only if for all $\varepsilon > 0$, there exists infinitely many terms of $\{a_n\}$ in $(L - \varepsilon, L + \varepsilon)$ but only finitely many terms of $\{a_n\}$ with $a_n > L + \varepsilon$.

Proof (continued). Second, suppose $\{a_n\}$ is a bounded sequence such that for all $\varepsilon > 0$ there exists infinitely many terms of $\{a_n\}$ in $(L - \varepsilon, L + \varepsilon)$ but only finitely many terms of $\{a_n\}$ with $a_n > L + \varepsilon$. Since $(L - \varepsilon, L + \varepsilon)$ contains infinitely many terms of $\{a_n\}$, then by Theorem 2-11 L is a subsequential limit of $\{a_n\}$. Let $M > L$ and define $\varepsilon' = (M - L)/2$. See Kirkwood's Figure 2-7 below. Then $(M - \varepsilon', M + \varepsilon')$ contains only finitely many terms of $\{a_n\}$ (since every term in $(M - \varepsilon', M + \varepsilon')$ is greater than $L + \varepsilon'$). But this M not a subsequential limit of $\{a_n\}$ (again by Theorem 2-11). Therefore, the greatest subsequential limit must be L . That is, by Exercise 2.3.16, $\lim a_n = L$ as claimed. \square



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Theorem 2-18(a)

Theorem 2-18.

- (a) $\lim(a_n + b_n) \leq \lim a_n + \lim b_n$.

Proof. (a) Let $M = \lim a_n$, $L = \lim b_n$, and let $\varepsilon > 0$. By Theorem 2-17, there are a finite number of terms of $\{a_n\}$ greater than or equal to $L + \varepsilon/2$ and a finite number of terms of $\{b_n\}$ greater than or equal to $L + \varepsilon/2$. Then all terms of $\{a_n + b_n\}$, except possible those with the same index as one of the above mentioned terms of $\{a_n\}$ or $\{b_n\}$ (of which there is a finite number) satisfy

$$a_n + b_n < (K + \varepsilon/2) + (L + \varepsilon/2) = (K + L) + \varepsilon.$$

So by Theorem 2-17, $\lim(a_n + b_n) \leq K + L$. (We actually only use half of Theorem 2-17, since we only get that this is less than or equal to $K + L$ and not equal to $K + L$.) This is the claim. \square

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Theorem 2-19(a)

Theorem 2-19. Let f and g be bounded functions with the same domain. Then:

- (a) $\sup(f + g) \leq \sup(f) + \sup(g)$.

Proof. (a) Since $\sup(f + g) = \text{lub}(\mathcal{R}(f + g))$, then by Theorem 2-8 there exists a sequence $\{y_n\}$ of points $y_n \in \mathcal{R}(f + g)$ such that $\{y_n\} \rightarrow \sup(f + g)$. Now $y_n = (f + g)(x_n) = f(x_n) + g(x_n)$ for some $x_n \in \mathcal{D}(f + g)$. But $f(x_n) \leq \sup(f)$ and $g(x_n) \leq \sup(g)$ for all $n \in \mathbb{N}$, so

$$\sup(f + g) = \lim y_n = \lim(f(x_n) + g(x_n)) \leq \sup(f) + \sup(g),$$

as claimed. \square

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Exercise 2.3.13

Exercise 2.3.13. Let $\{a_n\}$ be a Cauchy sequence.

- (a) Then $\{a_n\}$ is bounded.
 (b) There is at least one subsequential limit for $\{a_n\}$.
 (c) There is no more than one subsequential limit of $\{a_n\}$.
 (d) $\{a_n\}$ converges.

Proof. (a) By the definition of a Cauchy sequence, for $\varepsilon = 1$ there exists positive $N(\varepsilon) \in \mathbb{R}$ such that for all $m, n > N(\varepsilon)$ we have $|a_n - a_m| < 1$. In particular, with $N_1 = \lceil N(\varepsilon) + 1 \rceil$ we have $|a_m - a_{N_1}| < 1$ for all $m > N_1$. So $|a_m| < |a_{N_1}| + 1$ for all $m > N_1$. Therefore, $M = \max\{|a_1|, |a_2|, \dots, |a_{N_1}|, |a_{N_1}| + 1\}$ is an upper bound for $\{a_n\}$ and $-M$ is a lower bound. \square

(b) Since $\{a_n\}$ is bounded by (a), Theorem 2-14 implies that $\{a_n\}$ has a convergent subsequence. \square

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Exercise 2.3.13 (continued 1)

Exercise 2.3.13. Let $\{a_n\}$ be a Cauchy sequence.

(c) There is no more than one subsequential limit of $\{a_n\}$.

(d) $\{a_n\}$ converges.

Proof (continued). (c) ASSUME $L \neq M$ are both subsequential limits of $\{a_n\}$. Let $\{a_{n_\ell}\} \rightarrow L$ and $\{a_{n_m}\} \rightarrow M$ where, say, $L < M$. Let $\varepsilon = (M - L)/3$. Then, by the definition of limit of a subsequence, there exists positive $N_L(\varepsilon) = N_L$ such that for all $\ell > N_L$ we have $a_{n_\ell} \in (L - \varepsilon, L + \varepsilon) = ((4L - M)/3, (2L + M)/3)$. Similarly, there exists $N_M(\varepsilon) = N_M$ such that for all $m > N_M$ we have $a_{n_m} \in (M - \varepsilon, M + \varepsilon) = ((L + 2M)/3, (4M - L)/3)$. Let N be any positive real number. Then there some $a_{n_\ell} \in ((4L - M)/3, (2L + M)/3)$ and there is some $a_{n_m} \in ((L + 2M)/3, (4M - L)/3)$ where $\ell, m > N$ (since there are infinitely many such a_{n_ℓ} and a_{n_m} by Theorem 2-11).

Exercise 2.3.13 (continued 2)

Exercise 2.3.13. Let $\{a_n\}$ be a Cauchy sequence.

(c) There is no more than one subsequential limit of $\{a_n\}$.

(d) $\{a_n\}$ converges.

Proof (continued). If we take $\varepsilon' = (L + 2M)/3 - (2L + M)/3 = (M - L)/3$, then there can be no positive real N such that for all $\ell, m > N$ we have $|a_{n_\ell} - a_{n_m}| < \varepsilon' = (M - L)/3$ (because of the a_{n_ℓ} and a_{n_m} just described). That is, $\{a_n\}$ is not Cauchy, a CONTRADICTION. So the assumption that Cauchy sequence $\{a_n\}$ has two distinct subsequential limits is false, and hence there is no more than one subsequential limit of $\{a_n\}$, as claimed. \square

(d) Since there is only one subsequential limit by part (c), then by Exercise 2.3.16 we have that both $\overline{\lim} a_n$ and $\underline{\lim} a_n$ equal this subsequential limit. Therefore, $\overline{\lim} a_n = \underline{\lim} a_n$ and hence by Corollary 2-17, the sequence converges, as claimed. \square

Exercise 2.3.14

Exercise 2.3.14. A convergent sequence is Cauchy.

Proof. Let $\varepsilon > 0$. If $\{a_n\}$ is convergent, then there exists positive $N(\varepsilon) \in \mathbb{R}$ such that for all $n > N(\varepsilon)$ we have $|a_n - L| < \varepsilon/2$. Let $n, m > N$. Then by the Triangle Inequality

$$\begin{aligned} |a_n - a_m| &= |a_n - L + L - a_m| = |(a_n - L) - (a_m - L)| \\ &\leq |a_n - L| + |a_m - L| = \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Therefore, by definition, $\{a_n\}$ is Cauchy, as claimed. \square