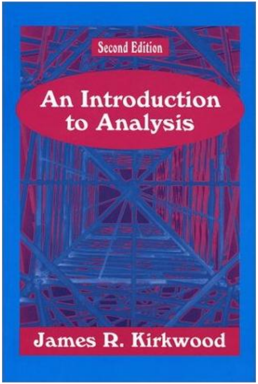
# Analysis 1

**Chapter 2. Sequences of Real Numbers**

2-3. The Bolzano-Weierstrass Theorem—Proofs of Theorems



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# Theorem 2-12

**Theorem 2-12. Bolzano-Weierstrass Theorem.**
Every bounded infinite set of real numbers has at least one limit point.

**Proof.** Let $A$ be a bounded set of real numbers. Since $A$ is bounded, then there is a positive $M \in \mathbb{R}$ such that $A \subset [-M, M] = A_0$. Then $A_0$ contains an infinite number of points in $A$. Cut $A_0$ into two closed subintervals of equal length, $[-M, 0]$ and $[0, M]$. Since $A_0$ contains an infinite number of points in $A$, the either $[-M, 0]$ or $[0, M]$ contains an infinite number of points in $A$; denote the set containing an infinite number of points in $A$ as $A_1$ and notice that the length of $A_1$ is $M$.

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# Theorem 2-12

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# Theorem 2-12 (continued 1)

**Theorem 2-12. Bolzano-Weierstrass Theorem.**
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**Proof (continued).** Each $A_k$ is a closed interval, so denote $A_k = [a_k, b_k]$. By construction, $A_k \supset A_{k+1}$ and $\lim_{k\to\infty}(b_k - a_k) = \lim_{k\to\infty} M/2^{k-1} = 0$ by Exercise 2.2.2(c). Then by Theorem 2-7, $\cap_{n=1}^{\infty} = \{p\}$ for some $p$. Next, we prove that $p$ is a limit point of set $A$.

Let $\varepsilon > 0$. Choose $N(\varepsilon) = N \in \mathbb{N}$ such that $M/2^{N-1} < \varepsilon$ (such $N(\varepsilon)$ exists by the definition of the limit of a sequence, since $\lim_{k\to\infty} M/2^{k-1} = 0$). Now $p \in A_N$, since $p$ is in $A_k$ for all $k \in \mathbb{N}$, and the length of $A_N$ is $M/2^{N-1}$. Therefore,

$$(p - \varepsilon, p + \varepsilon) \supset \left[p - \frac{M}{2^{N-1}}, p + \frac{M}{2^{N-1}}\right] \supset A_N.$$

This is illustrated below.

# Theorem 2-12 (continued 1)

**Theorem 2-12. Bolzano-Weierstrass Theorem.**
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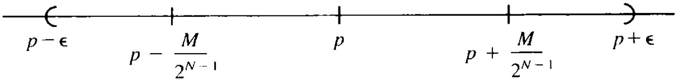
**Proof (continued).**




Kirkwood's Figure 2-6

Hence, $(p - \varepsilon, p + \varepsilon)$ contains infinitely many points of $A$ (since $A_N$ does). Since $\varepsilon > 0$ is arbitrary, then by the definition of limit point of a set, we have that $p$ is a limit point of set $A$. That is, set $A$ has at least one limit point, as claimed. □

# Theorem 2-14

**Theorem 2-14.** Every bounded sequence has a convergent subsequence.

**Proof.** Let $\{a_n\}$ be a bounded sequence. First, suppose there is a number $L$ for which an infinite number of terms of $\{a_n\}$ equal $L$, say $a_{n_1} = a_{n_2} = a_{n_3} = \cdots = L$ where $n_1 < n_2 < n_3 < \cdots$. Then the constant subsequence $\{a_{n_k}\} = \{a_{n_1}, a_{n_2}, a_{n_3}, \ldots\}$ converges to $L$.

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If there is no such $L$ repeated an infinite number of times in $\{a_n\}$, then $\{a_n\}$ but contain an infinite number of different terms. Let set $A$ be the set of terms in $\{a_n\}$. Then $A$ is an infinite set and, by hypothesis, it is bounded. By the Bolzano-Weierstrass Theorem (Theorem 2-12), there is a limit point $p$ of set $A$. So by the definition of limit point of a set, for all $\varepsilon > 0$ the interval $(p - \varepsilon, p + \varepsilon)$ contains an infinite number of points in set $A$. That is, $(p - \varepsilon, p + \varepsilon)$ contains an infinite number of terms of $\{a_n\}$. Then by Theorem 2-11, $p$ is a subsequential limit of $\{a_n\}$; that is, bounded sequence $\{a_n\}$ has a convergent subsequence as claimed. □

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# Theorem 2-15

**Theorem 2-15.**

(a) A sequence that is unbounded above has a subsequence that diverges to $+\infty$.

(b) A sequence that is unbounded below has a subsequence that diverges to $-\infty$.

**Proof.** **(a)** Let $\{a_n\}$ be sequence that is unbounded above. We give an inductive construction of a subsequence of $\{a_n\}$ that diverges to $\infty$, as Kirkwood does (we could also give a recursive construction of the subsequence).

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Let $P(k)$ be the statement that there is a positive integer $n_k > n_{k-1}$ such that $a_{n_k} > k$. For the Base Case observe that, since $\{a_n\}$ is unbounded above, there is a term greater than 1; denote such a term as $a_{n_1}$. Then $a_{n_1} > 1$ and $P(1)$ is true. For the Induction Hypothesis, suppose that $P(k)$ holds for all $k \in \mathbb{N}$ with $k \leq \ell$.

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(a) A sequence that is unbounded above has a subsequence that diverges to $+\infty$.

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**Proof (continued).** For the Induction Step, we consider $n = \ell + 1$. Notice that there are infinitely many terms of $\{a_n\}$ greater than $\ell + 1$, for if there were not then we could choose the last such term greater than $\ell + 1$, say $a_m$, and then the sequence would be bounded above by $\max\{a_1, a_2, \ldots, a_m, \ell + 1\}$, contradicting the hypothesized unbounded above property of $\{a_n\}$. So there is some term $a_{n_{\ell+1}}$ in $\{a_n\}$ where $n_{\ell+1} > n_\ell$ and $a_{\ell+1} > \ell + 1$. That is, $P(k)$ holds for $k = \ell + 1$ and the Induction Step is established. So, by the Induction Principle, $P(k)$ holds for all $k \in \mathbb{N}$. Therefore, there is a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $a_{n_k} > k$ for each $k \in \mathbb{N}$. By the definition of diverge to infinity, subsequence $\{a_{n_k}\}$ diverges to $\infty$, as claimed.

# Theorem 2-15 (continued)

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# Exercise 2.3.16(a)

**Exercise 2.3.16(a).** Let $\{a_n\}$ be a sequence. Then $\limsup a_n = \overline{\lim}\, a_n$ is a subsequential limit of $\{a_n\}$.

**Proof.** First, suppose $\overline{\lim}\, a_n = +\infty$. Let $P$ be the set of (real number) subsequential limits of $\{a_n\}$. Then $P$ is not bounded above (or else, by the Axiom of Completeness, $P$ would have a real number lub). For each $k \in \mathbb{N}$, there must be infinitely many terms of $\{a_n\}$ greater than $k+1$, for if there were not then we could choose the last such term greater than $k$, say $a_m$, and then the sequence would be bounded above by $\max\{a_1, a_2, \ldots, a_m, k+1\}$, contradicting fact that $P$ is not bounded above.

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# Exercise 2.3.16(a) (continued 1)

**Exercise 2.3.16(a).** Let $\{a_n\}$ be a sequence. Then $\limsup a_n = \overline{\lim}\, a_n$ is a subsequential limit of $\{a_n\}$.

**Proof (continued).** Second, suppose $\overline{\lim}\, a_n = -\infty$. Let $P$ be the set of (real number) subsequential limits of $\{a_n\}$. Set $P$ must be bounded above, or else $\overline{\lim}\, a_n = +\infty$ by the argument given above. If $P$ is bounded below then $P$ has a real number glb by the Axiom of Completeness and, since $\text{glb}(P) \leq \text{lub}(P)$, we cannot have $\text{lub}(P) = \overline{\lim}\, a_n = -\infty$. Hence $P$ cannot be bounded below and for an real number $-K-1$ there are infinitely many terms of $\{a_n\}$ less than $-K-1$. For $k=1$, there is some term of $\{a_n\}$ less than $-1$; denote it as $a_{n_1}$. Recursively define $\{a_{n_k}\}$ by letting $a_{n_{k+1}}$ be less than $-k-1$ and $n_{k+1} > n_k$ (such $a_{n_{k+1}}$ exists since there are infinitely many terms of $\{a_n\}$ less than $-k-1$). Then for any number $K$, there is $k \in \mathbb{N}$ with $-k \leq K$ (by the Archimedean Principle and Theorem 1-7(d)), so that $\{a_{n_k}\} \to -\infty$ (by the definition of a sequence diverges to $-\infty$). That is, $\overline{\lim}\, a_n = -\infty$ is a subsequential limit of $\{a_n\}$, as claimed.

# Exercise 2.3.16(a) (continued 1)

**Exercise 2.3.16(a).** Let $\{a_n\}$ be a sequence. Then $\limsup a_n = \overline{\lim}\, a_n$ is a subsequential limit of $\{a_n\}$.

**Proof (continued).** Second, suppose $\overline{\lim}\, a_n = -\infty$. Let $P$ be the set of (real number) subsequential limits of $\{a_n\}$. Set $P$ must be bounded above, or else $\overline{\lim}\, a_n = +\infty$ by the argument given above. If $P$ is bounded below then $P$ has a real number glb by the Axiom of Completeness and, since $\mathrm{glb}(P) \le \mathrm{lub}(P)$, we cannot have $\mathrm{lub}(P) = \overline{\lim}\, a_n = -\infty$. Hence $P$ cannot be bounded below and for an real number $-K-1$ there are infinitely many terms of $\{a_n\}$ less than $-K-1$. For $k=1$, there is some term of $\{a_n\}$ less than $-1$; denote it as $a_{n_1}$. Recursively define $\{a_{n_k}\}$ by letting $a_{n_{k+1}}$ be less than $-k-1$ and $n_{k+1} > n_k$ (such $a_{n_{k+1}}$ exists since there are infinitely many terms of $\{a_n\}$ less than $-k-1$). Then for any number $K$, there is $k \in \mathbb{N}$ with $-k \le K$ (by the Archimedean Principle and Theorem 1-7(d)), so that $\{a_{n_k}\} \to -\infty$ (by the definition of a sequence diverges to $-\infty$). That is, $\overline{\lim}\, a_n = -\infty$ is a subsequential limit of $\{a_n\}$, as claimed.

# Exercise 2.3.16(a) (continued 2)

**Exercise 2.3.16(a).** Let $\{a_n\}$ be a sequence. Then $\limsup a_n = \overline{\lim}\, a_n$ is a subsequential limit of $\{a_n\}$.

**Proof (continued).** Suppose $\overline{\lim}\, a_n = b \in \mathbb{R}$. Let $P$ be the set of (real number) subsequential limits of $\{a_n\}$. We will construct a subsequence which converges to $b$. Since (by definition of lim sup) $b = \operatorname{lub} P$ then for $\varepsilon = 1$ there exists $b_1 \in P$ such that $|b - b_1| < 1/2$ by Theorem 1-15(a). Since $b_1$ is a subsequential limit of of $\{a_n\}$, then there is term $a_{n_1}$ of $\{a_n\}$ such that $|b_1 - a_{n_1}| < 1/2$ by Theorem 2-11. Then

$$|b - a_{n_1}| = |b - b_1 + b_1 - a_{n_1}| \leq |b - b_1| + |b_1 - a_{n_1}| < 1/2 + 1/2 = 1.$$

We now create subsequence $a_{n_k}$ recursively. Let $k \in \mathbb{N}$ with $k > 1$. Consider $\varepsilon = 1/k$ and choose $b_k \in P$ such that $|b - b_k| < 1/(2k)$ (which can be done by Theorem 1-15(a)). There is a term $a_{n_k}$ of $\{a_n\}$ such that $|b_k - a_{n_k}| < 1/(2k)$ and $n_k > n_{k-1}$ (which can be done by Corollary 2-11).

# Exercise 2.3.16(a) (continued 2)

**Exercise 2.3.16(a).** Let $\{a_n\}$ be a sequence. Then $\limsup a_n = \overline{\lim}\, a_n$ is a subsequential limit of $\{a_n\}$.

**Proof (continued).** Suppose $\overline{\lim}\, a_n = b \in \mathbb{R}$. Let $P$ be the set of (real number) subsequential limits of $\{a_n\}$. We will construct a subsequence which converges to $b$. Since (by definition of lim sup) $b = \operatorname{lub} P$ then for $\varepsilon = 1$ there exists $b_1 \in P$ such that $|b - b_1| < 1/2$ by Theorem 1-15(a). Since $b_1$ is a subsequential limit of of $\{a_n\}$, then there is term $a_{n_1}$ of $\{a_n\}$ such that $|b_1 - a_{n_1}| < 1/2$ by Theorem 2-11. Then

$$|b - a_{n_1}| = |b - b_1 + b_1 - a_{n_1}| \le |b - b_1| + |b_1 - a_{n_1}| < 1/2 + 1/2 = 1.$$

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# Exercise 2.3.16(a) (continued 3)

**Exercise 2.3.16(a).** Let $\{a_n\}$ be a sequence. Then $\limsup a_n = \overline{\lim}\, a_n$ is a subsequential limit of $\{a_n\}$.

**Proof (continued).** Then

$$|b-a_{n_k}| = |b-b_k+b_k-a_{n_k}| \le |b-b_k|+|b_k-a_{n_k}| < 1/(2k)+1/(2k) = 1/k.$$

For any $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that $1/k < \varepsilon$ by the Archimedean Principle (Theorem 1-18). By the definition of limit of a sequence $\{a_{n_k}\} \to b$, so that $b$ is a subsequential limit of $\{a_n\}$. That is, $b = \overline{\lim}\, a_n$ is a subsequential limit of $\{a_n\}$, as claimed. □

# Theorem 2-17(a)

**Theorem 2-17.** Let $\{a_n\}$ be a bounded sequence. Then

(a) $\overline{\lim}\, a_n = L$ if and only if for all $\varepsilon > 0$, there exists infinitely many terms of $\{a_n\}$ in $(L - \varepsilon, L + \varepsilon)$ but only finitely many terms of $\{a_n\}$ with $a_n > L + \varepsilon$.

**Proof. (a)** First, let $\overline{\lim}\, a_n = L$ and let $\varepsilon > 0$ be given. Since $L$ is a subsequential limit of $\{a_n\}$ by Exercise 2.3.16(a), then by Theorem 2-11 there are infinitely many terms of $\{a_n\}$ in $(L - \varepsilon, L + \varepsilon)$. ASSUME there are an infinite number of terms of $\{a\}$ with $a_n > L + \varepsilon$. Since $\{a_n\}$ is bounded above, say by $M$, there are infinitely many terms of $\{a_n\}$ between $L + \varepsilon$ and $M$.

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**Theorem 2-17.** Let $\{a_n\}$ be a bounded sequence. Then

(a) $\overline{\lim}\, a_n = L$ if and only if for all $\varepsilon > 0$, there exists infinitely many terms of $\{a_n\}$ in $(L - \varepsilon, L + \varepsilon)$ but only finitely many terms of $\{a_n\}$ with $a_n > L + \varepsilon$.

**Proof. (a)** First, let $\overline{\lim}\, a_n = L$ and let $\varepsilon > 0$ be given. Since $L$ is a subsequential limit of $\{a_n\}$ by Exercise 2.3.16(a), then by Theorem 2-11 there are infinitely many terms of $\{a_n\}$ in $(L - \varepsilon, L + \varepsilon)$. ASSUME there are an infinite number of terms of $\{a\}$ with $a_n > L + \varepsilon$. Since $\{a_n\}$ is bounded above, say by $M$, there are infinitely many terms of $\{a_n\}$ between $L + \varepsilon$ and $M$. By Exercise 2.3.5, there is a subsequence $\{a_{n_k}\}$ with a limit at least as large as $L + \varepsilon$. But this means that $L + \varepsilon$ is a subsequential limit greater than $\overline{\lim}\, a_n = L$, a CONTRADICTION since $L$ is an upper bound for the set of subsequential limits. This contradiction shows that the assumption that there are an infinite number of terms of $\{a\}$ with $a_n > L + \varepsilon$ is false. Hence, there are only a finite number of terms of $\{a_n\}$ with $a_n > L + \varepsilon$, as claimed.

# Theorem 2-17(a) (continued)

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**Proof (continued).** Second, suppose $\{a_n\}$ is a bounded sequence such that for all $\varepsilon > 0$ there exists infinitely many terms of $\{a_n\}$ in $(L-\varepsilon, L+\varepsilon)$ but only finitely many terms of $\{a_n\}$ with $a_n > L+\varepsilon$. Since $(L-\varepsilon, L+\varepsilon)$ contains infinitely many terms of $\{a_n\}$, then by Theorem 2-11 $L$ is a subsequential limit of $\{a_n\}$. Let $M > L$ and define $\varepsilon' = (M-L)/2$. See Kirkwood's Figure 2-7 below. Then $(M-\varepsilon, M+\varepsilon)$ contains only finitely many terms of $\{a_n\}$ (since every term in $(M-\varepsilon', M+\varepsilon')$ is greater than $L+\varepsilon'$). But this $M$ not a subsequential limit of $\{a_n\}$ (again by Theorem 2-11). Therefore, the greatest subsequential limit must be $L$. That is, by Exercise 2.3.16, $\overline{\lim}\, a_n = L$ as claimed. □

# Theorem 2-17(a) (continued)

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(a) $\overline{\lim}\, a_n = L$ if and only if for all $\varepsilon > 0$, there exists infinitely many terms of $\{a_n\}$ in $(L-\varepsilon, L+\varepsilon)$ but only finitely many terms of $\{a_n\}$ with $a_n > L+\varepsilon$.

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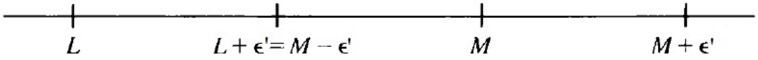

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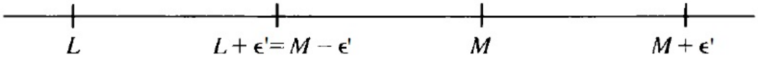

# Theorem 2-18(a)

**Theorem 2-18.**

(a) $\overline{\lim}(a_n + b_n) \leq \overline{\lim}\, a_n + \overline{\lim}\, b_n$.

**Proof. (a)** Let $M = \overline{\lim}\, a_n$, $L = \overline{\lim}\, b_n$, and let $\varepsilon > 0$. By Theorem 2-17, there are a finite number of terms of $\{a_n\}$ greater than or equal to $L + \varepsilon/2$ and a finite number of terms of $\{b_n\}$ greater then or equal to $L + \varepsilon/2$. Then all terms of $\{a_n + b_n\}$, except possible those with the same index as one of the above mentioned terms of $\{a_n\}$ or $\{b_n\}$ (of which there is a finite number) satisfy

$$a_n + b_n < (K + \varepsilon/2) + (L + \varepsilon/2) = (K + L) + \varepsilon.$$

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# Theorem 2-19(a)

**Theorem 2-19.** Let $f$ and $g$ be bounded functions with the same domain. Then:

(a) $\sup(f+g) \leq \sup(f) + \sup(g)$.

**Proof.** **(a)** Since $\sup(f+g) = \text{lub}(\mathcal{R}(f+g))$, then by Theorem 2-8 there exists a sequence $\{y_n\}$ of points $y_n \in \mathcal{R}(f+g)$ such that $\{y_n\} \to \sup(f+g)$. Now $y_n = (f+g)(x_n) - f(x_n) + g(x_n)$ for some $x_n \in \mathcal{D}(f+g)$. But $f(x_n) \leq \sup(f)$ and $g(x_n) \leq \sup(g)$ for all $n \in \mathbb{N}$, so

$$\sup(f+g) = \lim y_n = \lim(f(x_n) + g(x_n)) \leq \sup(f) + \sup(g),$$

as claimed. $\square$

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$$\sup(f+g) = \lim y_n = \lim(f(x_n) + g(x_n)) \leq \sup(f) + \sup(g),$$

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# Exercise 2.3.13

**Exercise 2.3.13.** Let $\{a_n\}$ be a Cauchy sequence.

(a) Then $\{a_n\}$ is bounded.

(b) There is at least one subsequential limit for $\{a_n\}$.

(c) There is no more than one subsequential limit of $\{a_n\}$.

(d) $\{a_n\}$ converges.

**Proof. (a)** By the definition of a Cauchy sequence, for $\varepsilon = 1$ there exists positive $N(\varepsilon) =\in \mathbb{R}$ such that for all $m, n > N(\varepsilon)$ we have $|a_n - a_m| < 1$. In particular, with $N_1 = \lceil N(\varepsilon) + 1 \rceil$ we have $|a_m - a_{N_1}| < 1$ for all $m > N_1$. So $|a_m| < |a_{N_1}| + 1$ for all $m > N_1$. Therefore, $M = \max\{|a_1|, |a_2|, \ldots, |a_{N_1}|, |a_{N_1}| + 1\}$ is an upper bound for $\{a_n\}$ and $-M$ is a lower bound. □

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**(b)** Since $\{a_n\}$ is bounded by (a), Theorem 2-14 implies that $\{a_n\}$ has a convergent subsequence. □

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# Exercise 2.3.13 (continued 1)

**Exercise 2.3.13.** Let $\{a_n\}$ be a Cauchy sequence.

(c) There is no more than one subsequential limit of $\{a_n\}$.

(d) $\{a_n\}$ converges.

**Proof (continued). (c)** ASSUME $L \neq M$ are both subsequential limits of $\{a_n\}$. Let $\{a_{n_\ell}\} \to L$ and $\{a_{n_m}\} \to M$ where, say, $L < M$. Let $\varepsilon = (M - L)/3$. Then, by the definition of limit of a subsequence, there exists positive $N_L(\varepsilon) = N_L$ such that for all $\ell > N_L$ we have $a_{n_\ell} \in (L - \varepsilon, L + \varepsilon) = ((4L - M)/3, (2L + M)/3)$. Similarly, there exists $N_M(\varepsilon) = N_M$ such that for all $m > N_M$ we have $a_{n_m} \in (M - \varepsilon, M + \varepsilon) = ((L + 2M)/3, (4M - L)/3)$. Let $N$ be any positive real number. Then there some $a_{n_\ell} \in ((4L - M)/3, (2L + M)/3)$ and there is some $a_{n_m} \in ((L + 2M)/3, (4M - L)/3)$ where $\ell, m > N$ (since there are infinitely many such $a_{n_\ell}$ and $a_{n_m}$ by Theorem 2-11).

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# Exercise 2.3.13 (continued 2)

**Exercise 2.3.13.** Let $\{a_n\}$ be a Cauchy sequence.

(c) There is no more than one subsequential limit of $\{a_n\}$.

(d) $\{a_n\}$ converges.

**Proof (continued).** If we take $\varepsilon' = (L+2M)/3 - (2L+M)/3 = (M-L)/3$, then there can be no positive real $N$ such that for all $\ell, m > N$ we have $|a_{n_\ell} - a_{n_m}| < \varepsilon' = (M-L)/3$ (because of the $a_{n_\ell}$ and $a_{n_m}$ just described). That is, $\{a_n\}$ is not Cauchy, a CONTRADICTION. So the assumption that Cauchy sequence $\{a_n\}$ has two distinct subsequential limits is false, and hence there is no more than one subsequential limit of $\{a_n\}$, as claimed. □

**(d)** Since there is only one subsequential limit by part (c), then by Exercise 2.3.16 we have that both $\overline{\lim}\, a_n$ and $\underline{\lim}\, a_n$ equal this subsequential limit. Therefore, $\overline{\lim}\, a_n = \underline{\lim}\, a_n$ and hence by Corollary 2-17, the sequence converges, as claimed. □

# Exercise 2.3.13 (continued 2)

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# Exercise 2.3.14

**Exercise 2.3.14.** A convergent sequence is Cauchy.

**Proof.** Let $\varepsilon > 0$. If $\{a_n\}$ is convergent, then there exists positive $N(\varepsilon) \in \mathbb{R}$ such that for all $n > N(\varepsilon)$ we have $|a_n - L| < \varepsilon/2$. Let $n, m > N$. Then by the Triangle Inequality

$$|a_n - a_m| = |a_n - L + L - a_m| = |(a_n - L) - (a_m - L)|$$

$$\leq |a_n - L| + |a_m - L| = \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Therefore, by definition, $\{a_n\}$ is Cauchy, as claimed. □

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