Analysis 1

Chapter 3. Topology of the Real Numbers

3-1. Topology of the Real Numbers—Proofs of Theorems

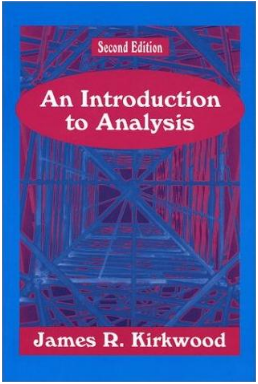

Table of contents

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Theorem 3-1. The intervals (a, b), (a, ∞), and $(-\infty, a)$ are open sets.

Proof. For interval $U = (a, b)$, let $x \in (a, b)$. Then $a < x < b$. Define $\delta(x) = \min\{x - a, b - x\}$ (see Kirkwood's Figure 3-1 below). Then $a = a + x - x = x - (x - a) \leq x - \delta(x)$ (since $\delta(x) \leq x - a$ or $-\delta(x) \geq -(x - a)$ or $-(x - a) \leq -\delta(x)$), and (since $\delta(x) \leq b - x$) $x + \delta(x) \leq x + (b - x) = b$, so that $a \leq x - \delta(x) < x + \delta(x) < b$. Therefore $(x - \delta(x), x + \delta(x)) \subset (a, b) = U$ and, by the definition of open set, (a, b) is open.

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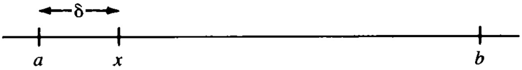

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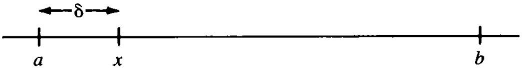

Theorem 3-1 (continued)

Theorem 3-1. The intervals (a, b), (a, ∞), and $(-\infty, a)$ are open sets.

Proof (continued). For interval $U = (a, \infty)$, let $x \in (a, \infty)$. Then $a < x$. Define $\delta(x) = x - a$. Then $a = a + x - x = x - (x - a) = x - \delta(x)$, and $x + \delta(x) = x + (x - a) = 2x - a < \infty$, so that $a = x - \delta(x) < x + \delta(x) < \infty$. Therefore $(x - \delta(x), x + \delta(x)) \subset (a, \infty) = U$ and, by the definition of open set, (a, ∞) is open.

For interval $U = (-\infty, a)$, let $x \in (-\infty, a)$. Then $x < a$. Define $\delta(x) = a - x$. Then $-\infty < 2x - a = x - (a - x) = x - \delta(x)$ and $x + \delta(x) = x + (a - x) = a$, so that $-\infty < x - \delta(x) < x + \delta(x) = a$. Therefore $(x - \delta(x), x + \delta(x)) \subset (-\infty, a) = U$ and, by the definition of open set, $(-\infty, a)$ is open. □

Theorem 3-1 (continued)

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Proof (continued). For interval $U = (a, \infty)$, let $x \in (a, \infty)$. Then $a < x$. Define $\delta(x) = x - a$. Then $a = a + x - x = x - (x - a) = x - \delta(x)$, and $x + \delta(x) = x + (x - a) = 2x - a < \infty$, so that $a = x - \delta(x) < x + \delta(x) < \infty$. Therefore $(x - \delta(x), x + \delta(x)) \subset (a, \infty) = U$ and, by the definition of open set, (a, ∞) is open.

For interval $U = (-\infty, a)$, let $x \in (-\infty, a)$. Then $x < a$. Define $\delta(x) = a - x$. Then $-\infty < 2x - a = x - (a - x) = x - \delta(x)$ and $x + \delta(x) = x + (a - x) = a$, so that $-\infty < x - \delta(x) < x + \delta(x) = a$. Therefore $(x - \delta(x), x + \delta(x)) \subset (-\infty, a) = U$ and, by the definition of open set, $(-\infty, a)$ is open. □

Theorem 3-2

Theorem 3-2. The open sets satisfy:

(a) If $\{U_1, U_2, \ldots, U_n\}$ is a *finite* collection of open sets, then $\cap_{k=1}^n U_k$ is an open set.

(b) If $\{U_\alpha\}$ is *any* collection (finite, infinite, countable, or uncountable) of open sets, then $\cup_\alpha U_\alpha$ is an open set.

Proof. (a) Let $U_1, U_2, \ldots, U_n$ be open sets and let $x \in \cap_{k=1}^n U_k$. Since U_k is open and $x \in U_k$, then there is $\delta_k(x) > 0$ such that $(x - \delta_k(x), x + \delta_k(x)) \subset U_k$, and this holds for every $k = 1, 2, \ldots, n$. Define $\delta(x) = \min\{\delta_1(x), \delta_2(x), \ldots, \delta_n(x)\} = \min_{1 \le k \le n}\{\delta_k(x)\}$. Notice that since $\delta(x)$ is a minimum over a finite set of positive numbers so that $\delta(x)$ is positive (we could not do this for an infinite collection of $\delta_k(x)$'s).

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Proof. (a) Let $U_1, U_2, \ldots, U_n$ be open sets and let $x \in \cap_{k=1}^n U_k$. Since U_k is open and $x \in U_k$, then there is $\delta_k(x) > 0$ such that $(x - \delta_k(x), x + \delta_k(x)) \subset U_k$, and this holds for every $k = 1, 2, \ldots, n$. Define $\delta(x) = \min\{\delta_1(x), \delta_2(x), \ldots, \delta_n(x)\} = \min_{1 \le k \le n}\{\delta_k(x)\}$. Notice that since $\delta(x)$ is a minimum over a finite set of positive numbers so that $\delta(x)$ is positive (we could not do this for an infinite collection of $\delta_k(x)$'s). Then $(x - \delta(x), x + \delta(x)) \subset (x - \delta_k(x), x + \delta_k(x)) \subset U_k$ for each $k = 1, 2, \ldots, n$. Therefore $(x - \delta(x), x + \delta(x)) \subset \cap_{k=1}^n U_k$. That is, by the definition of open set, $\cap_{k=1}^n U_k$ is an open set, as claimed.

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Proof. (a) Let $U_1, U_2, \ldots, U_n$ be open sets and let $x \in \cap_{k=1}^{n} U_k$. Since U_k is open and $x \in U_k$, then there is $\delta_k(x) > 0$ such that $(x - \delta_k(x), x + \delta_k(x)) \subset U_k$, and this holds for every $k = 1, 2, \ldots, n$. Define $\delta(x) = \min\{\delta_1(x), \delta_2(x), \ldots, \delta_n(x)\} = \min_{1 \le k \le n}\{\delta_k(x)\}$. Notice that since $\delta(x)$ is a minimum over a finite set of positive numbers so that $\delta(x)$ is positive (we could not do this for an infinite collection of $\delta_k(x)$'s). Then $(x - \delta(x), x + \delta(x)) \subset (x - \delta_k(x), x + \delta_k(x)) \subset U_k$ for each $k = 1, 2, \ldots, n$. Therefore $(x - \delta(x), x + \delta(x)) \subset \cap_{k=1}^{n} U_k$. That is, by the definition of open set, $\cap_{k=1}^{n} U_k$ is an open set, as claimed.

Theorem 3-2 (continued)

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Proof (continued). (b) Let $\{U_\alpha\}$ is any collection of open sets and let $x \in \cup_\alpha U_\alpha$. Then for some α', we have $x \in U_{\alpha'}$. Since $U_{\alpha'}$ is open and $x \in U_{\alpha'}$, then there is $\delta(x) > 0$ such that $(x - \delta(x), x + \delta(x)) \subset U_{\alpha'}$. Then $(x - \delta(x), x + \delta(x)) \subset \cup_\alpha U_\alpha$. That is, by the definition of open set, $\cup_\alpha U_\alpha$ is an open set, as claimed. □

Theorem 3-6

Theorem 3-6. A set is closed if and only if it contains all of its boundary points.

Proof. First, let A be a closed set and let x be a boundary point of A. ASSUME $x \notin A$. Then $x \in A^c$ and, since A is closed, A^c is open. So, by the definition of open set, there is $\delta > 0$ such that $(x - \delta, x + \delta) \subset A^c$. But then $(x - \delta, x + \delta)$ contains no points of set A itself, CONTRADICTING the fact that x is a boundary point of A (see the definition of boundary point). This contradiction shows that the assumption that $x \notin A$ is false. That is, every boundary point of closed set A must be an element of A.

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Secondly, suppose A contains all of its boundary points. Let $x \in A^c$. Then $x \notin A$ so that x is not a boundary point of A. So (by the negation of the definition of boundary point) there is $\delta > 0$ such that $(x - \delta, x + \delta)$ contains no points of A. That is, $(x - \delta, x + \delta) \subset A^c$. Since x is an arbitrary point of A^c, then A^c is open and hence, as claimed, A is closed. □

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Theorem 3-7

Theorem 3-7. For $A \subset \mathbb{R}$, $\overline{A}$ is closed.

Proof. Consider $\overline{A}^c$ and let $x \in \overline{A}^c$. Since $x \notin A$ and x is not a boundary point of A, then (by the negation of the definition of boundary point; see Note 3.1.H) there is some $\delta > 0$ such that $(x - \delta, x + \delta)$ contains no points of A; that is, $(x - \delta, x + \delta) \subset A^c$. (We still need to show that $(x - \delta, x + \delta) \subset \overline{A}^c$.)

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For any $y \in (x - \delta, x + \delta)$, define $\delta' = \min\{y - (x - \delta), (x + \delta) - y\}$. Then $(y - \delta', y + \delta') \subset (x - \delta, x + \delta)$ contains no points of A, and hence y is not a boundary point of A. Since y is an arbitrary point of $(x - \delta, x + \delta)$, then no points of $(x - \delta, x + \delta)$ are boundary points of A. We now have that $(x - \delta, x + \delta)$ contains no points of A and no boundary points of A, so that $(x - \delta, x + \delta) \subset \overline{A}^c$. Since x is an arbitrary point of $\overline{A}^c$, then (by the definition of open set) $\overline{A}^c$ is open. That is, $\overline{A}$ is closed, as claimed. □

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Theorem 3-8

Theorem 3-8. Let $\{A_1, A_2, \ldots\}$ be a countable collection of nonempty closed bounded sets of real numbers such that $A_i \supset A_j$ for $i \leq j$. Then $\cap A_i \neq \varnothing$.

Proof. Since each A_i is nonempty, then there is some $a_i \in A_i$ for each $i \in \mathbb{N}$. Since $A_i \supset A_j$ for $i \leq j$ by hypothesis, the all sets are subsets of A_1 and sequence $\{a_i\}$ forms a subset of A_1. Since A_1 is bounded, then sequence $\{a_i\}$ is bounded. By Theorem 2-14 there is a subsequence $\{a_{i_k}\}$ of $\{a_i\}$ that is convergent. Let $\{a_{i_k}\} \to p$.

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Let $\varepsilon > 0$. Since $\{a_{i_k}\} \to p$ then, by the definition of limit of a sequence, there is $N \in \mathbb{R}$ such that for all $k > N$ we have $|p - a_{i_k}| < \varepsilon$, or $a_{i_k} \in (p - \varepsilon, p + \varepsilon)$. Since the sets are nested, then for any $a_{i_k} \in (p - \varepsilon, p + \varepsilon)$, we have $a_{i_k} \in A_i$ for all $i \leq i_k$. Therefore $(p - \varepsilon, p + \varepsilon)$ contains a point of every A_i.

Theorem 3-8

Theorem 3-8. Let $\{A_1, A_2, \ldots\}$ be a countable collection of nonempty closed bounded sets of real numbers such that $A_i \supset A_j$ for $i \leq j$. Then $\cap A_i \neq \varnothing$.

Proof. Since each A_i is nonempty, then there is some $a_i \in A_i$ for each $i \in \mathbb{N}$. Since $A_i \supset A_j$ for $i \leq j$ by hypothesis, the all sets are subsets of A_1 and sequence $\{a_i\}$ forms a subset of A_1. Since A_1 is bounded, then sequence $\{a_i\}$ is bounded. By Theorem 2-14 there is a subsequence $\{a_{i_k}\}$ of $\{a_i\}$ that is convergent. Let $\{a_{i_k}\} \to p$.

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Theorem 3-8 (continued)

Theorem 3-8. Let $\{A_1, A_2, \ldots\}$ be a countable collection of nonempty closed bounded sets of real numbers such that $A_i \supset A_j$ for $i \leq j$. Then $\cap A_i \neq \varnothing$.

Proof (continued). If $p \notin A_i$ then $(p - \varepsilon, p + \varepsilon)$ contains both a point in A_i and a point not in A_i (the point not in A_i is p in this case); that is, p is a limit point of A_i. So for every $i \in \mathbb{N}$, either $p \in A_i$ or p is a limit point of A_i. Since each A_i is closed then, by Corollary 3-6(a), then each A_i contains its limit points. Therefore, we must have $p \in A_i$ for all $i \in \mathbb{N}$. Hence, $p \in \cap A_i$ and $\cap A_i \neq \varnothing$, as claimed. □

Corollary 3-8

Corollary 3-8. Let $\{A_1, A_2, \ldots\}$ be a countable collection of closed bounded sets of real numbers such that $A_i \supset A_j$ if $i < j$. If $\cap_{i=1}^{\infty} A_i = \varnothing$ then $\cap_{i=1}^{N} A_i = \varnothing$ for some $N \in \mathbb{N}$.

Proof. By Theorem 3-8, if all the sets are nonempty then $\cap_{i=1}^{\infty} A_i \neq \varnothing$. Since by hypothesis $\cap_{i=1}^{\infty} A_i = \varnothing$, then there must be some $A_N = \varnothing$. (In fact, because the sets are nested, we must then have $A_i \varnothing$ for all $i \geq N$.) So we have $\cap_{i=1}^{N} A_i = \varnothing$, as claimed. □

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Proof. By Theorem 3-8, if all the sets are nonempty then $\cap_{i=1}^{\infty} A_i \neq \varnothing$. Since by hypothesis $\cap_{i=1}^{\infty} A_i = \varnothing$, then there must be some $A_N = \varnothing$. (In fact, because the sets are nested, we must then have $A_i \varnothing$ for all $i \geq N$.) So we have $\cap_{i=1}^{N} A_i = \varnothing$, as claimed. □

Theorem 3-10

Theorem 3-10. Heine-Borel Theorem.
If A is a closed and bounded set of real numbers, then A is compact.

Proof. Let A be a closed and bounded set of real numbers and $\{I_\alpha\}$ an open cover of A. By Theorem 3-9 (The Lindelöf Property), there is a countable open subcover of A, $\{I_1, I_2, \ldots\}$. Define the sets $J_n = \cup_{i=1}^{n} I_i$ for each $n \in \mathbb{N}$. Each J_n is open by Theorem 3-2(b). This is an *increasing* sequence of sets; that is, $J_n \subset J_{n+1}$ for each $n \in \mathbb{N}$. Since $\cup_{n=1}^{\infty} J_n = \cup_{n=1}^{\infty} I_n$ then $\{J_1, J_2, \ldots\}$ is also a countable open cover of A.

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$$\varnothing = A \setminus (\cup_{i=1}^{\infty} J_i) = \cup_{i=1}^{\infty} (A \setminus J_i) = \cap_{i=1}^{\infty} K_i.$$

Theorem 3-10

Theorem 3-10. Heine-Borel Theorem.
If A is a closed and bounded set of real numbers, then A is compact.

Proof. Let A be a closed and bounded set of real numbers and $\{I_\alpha\}$ an open cover of A. By Theorem 3-9 (The Lindelöf Property), there is a countable open subcover of A, $\{I_1, I_2, \ldots\}$. Define the sets $J_n = \cup_{i=1}^n I_i$ for each $n \in \mathbb{N}$. Each J_n is open by Theorem 3-2(b). This is an *increasing* sequence of sets; that is, $J_n \subset J_{n+1}$ for each $n \in \mathbb{N}$. Since $\cup_{n=1}^\infty J_n = \cup_{n=1}^\infty I_n$ then $\{J_1, J_2, \ldots\}$ is also a countable open cover of A. Next, define $K_n = A \setminus J_n$ for each $n \in \mathbb{N}$. This is a *decreasing* sequence of sets; that is, $K_n \supset K_{n+1}$ for each $n \in \mathbb{N}$. Since A is closed and J_n is open, then each K_n is closed by Exercise 3.1.6(a). Since $\{J_1, J_2, \ldots\}$ is a cover of A, then $A \setminus (\cup_{i=1}^\infty J_i) = \varnothing$. By DeMorgan's Laws (Corollary 1-1 and Exercise 1.1.8):

$$\varnothing = A \setminus (\cup_{i=1}^\infty J_i) = \cup_{i=1}^\infty (A \setminus J_i) = \cap_{i=1}^\infty K_i.$$

Theorem 3-10

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If A is a closed and bounded set of real numbers, then A is compact.

Proof (continued). By Corollary 3-8 as applied to sets $\{K_1, K_2, \ldots\}$, there is $n \in \mathbb{N}$ such that $\cap_{i=1}^{N} K_i = \varnothing$. We now have

$$\varnothing = \cap_{i=1}^{N} K_i = \cap_{i=1}^{N} (A \setminus J_i) = A \setminus \left(\cup_{i=1}^{N} J_i \right) = A \setminus \left(\cup_{i=1}^{N} I_i \right).$$

Therefore, $\{I_1, I_2, \ldots, I_N\}$ is a finite open cover of A. Hence, by definition, A is compact as claimed. □

Theorem 3-11

Theorem 3-11. A set that is compact is closed and bounded.

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Suppose A is not closed. Then by Corollary 3-6(a) there is at least one limit point of A, say x, such that $x \notin A$. Let

$$I_n = (-\infty, x - 1/n) \cup (x + 1/n, \infty) \text{ for } n \in \mathbb{N}.$$

Then $\{I_n\}$ is an open cover of A. ASSUME there is a finite subcover of A. Then there is a largest value of the index, say N, in the subcover. The union of the elements of the subcover is $(-\infty, x - 1/N) \cup (x + 1/N, \infty)$.

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Theorem 3-11 (continued)

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Proof (continued). With $\varepsilon = 1/N$, the interval $(x - \varepsilon, x + \varepsilon) = (x - 1/N, x + 1/N)$ contains an element of A (since x is a limit point of A), but this point of A is not in the finite open cover, a CONTRADICTION to the fact that the finite subcover is a superset of set A. So the assumption that there is a finite subcover of A is false. That is, $\{I_n\}$ is an open cover of A without any finite subcover. So A is not compact. This establishes the contrapositive of the claim. □

Theorem 3-12

Theorem 3-12. A set $A \subset \mathbb{R}$ is compact if and only if every infinite set of points of A has a limit point in A.

Proof. First, let A be compact with $B \subset A$ where B is an infinite set. Since A is compact, then A is bounded by the Heine-Borel Theorem (Theorem 3-10). Since $B \subset A$, then set B is also bounded. Now by the Bolzano-Weierstrass Theorem (Theorem 2-12), set B has a limit point, say p.

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Now we need to prove that if every infinite set of points in A has a limit point in A, then A is compact. We prove the contrapositive: If A is not compact, then there is an infinite set of points in A that does not have a limit point in A.

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Theorem 3-12 (continued 1)

Proof (continued). Let set A not be compact. Then by Theorem 3-11, either A is not closed or A is not bounded. If A not not closed then (by Corollary 3-6(a)) there is a limit point of A that is not in A, say point x. By the definition of limit point, the interval $(x - 1/n, x + 1/n)$ contains some point of A distinct from x for each $n \in \mathbb{N}$. Choose one such point from each interval and denote it as x_n. ASSUME $\{x_1, x_2, \ldots\}$ contains only finitely many different points. Then there is some x_i that is closest to x, say x_N. With $\varepsilon' = |x - x_N|$, the interval $(x - \varepsilon', x + \varepsilon')$ contains no points of A, CONTRADICTING the fast that x is a limit point of A. So the assumption that $\{x_1, x_2, \ldots\}$ contains only finitely many different points is false, and so $B = \{x_1, x_2, \ldots, \}$ contains infinitely many points of A.

Theorem 3-12 (continued 1)

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Proof (continued). Let set A not be compact. Then by Theorem 3-11, either A is not closed or A is not bounded. If A not not closed then (by Corollary 3-6(a)) there is a limit point of A that is not in A, say point x. By the definition of limit point, the interval $(x - 1/n, x + 1/n)$ contains some point of A distinct from x for each $n \in \mathbb{N}$. Choose one such point from each interval and denote it as x_n. ASSUME $\{x_1, x_2, \ldots\}$ contains only finitely many different points. Then there is some x_i that is closest to x, say x_N. With $\varepsilon' = |x - x_N|$, the interval $(x - \varepsilon', x + \varepsilon')$ contains no points of A, CONTRADICTING the fast that x is a limit point of A. So the assumption that $\{x_1, x_2, \ldots\}$ contains only finitely many different points is false, and so $B = \{x_1, x_2, \ldots, \}$ contains infinitely many points of A. Since $1/n$ can be made arbitrarily small (by the Archimedean Principle, Theorem 1-18), then the sequence $\{x_i\} \to x$ and the set $\{x_1, x_2, \ldots\}$ has only one limit point, namely x. That is, $B = \{x_1, x_2, \ldots\}$ is an infinite set of points of A that does not have a limit point in A, as needed.

Theorem 3-12 (continued 2)

Theorem 3-12. A set $A \subset \mathbb{R}$ is compact if and only if every infinite set of points of A has a limit point in A.

Proof (continued). Finally, let set A not be bounded. Then for each $n \in \mathbb{N}$ there is some x_n is A such that $x_n \notin [-n, n]$. Define $B = \{x_n \mid n \in \mathbb{N}\}$, so that B is an infinite set of elements of A. Then for any $x \in \mathbb{R}$ and for any given $\varepsilon > 0$, the interval $(x - \varepsilon, x + \varepsilon)$ contains no x_n for $n > \max\{|x - \varepsilon|, |x + \varepsilon|\}$. That is, $(x - \varepsilon, x + \varepsilon)$ contains only finitely many points of B. So there are no limit points of infinite set B and hence there can be no limit point of B in set A. That is, $B = \{x_1, x_2, \ldots\}$ is an infinite set of points of A that does not have a limit point in A, as needed. □

Theorem 3-14

Theorem 3-14. A set of real numbers with more than one element is connected if and only if it is an interval.

Proof. Let A be a set of real numbers with more than one element, which is *not* an interval. By negating the definition of interval, we have that A is not an interval if there are $r, s \in A$ with $r < s$ such that there is a t with $r < t < s$ and $t \notin A$. Let $U = (-\infty, t)$ and $V = (t, \infty)$. Then U and V are disjoint open sets with $r \in U \cap A = (-\infty, t) \cap A \neq \varnothing$, $s \in V \cap A = (t, \infty) \cap A \neq \varnothing$, and

$$[U \cap A] \cup [V \cap A] = [(-\infty, t) \cap A] \cup [(t, \infty) \cap A] = A.$$

Hence, by definition, A is not connected. That is, if A is a set of real numbers with more than one element which is not an interval then A is not connected. In other words (i.e., the contrapositive of this statement) if A is a connected set with more than one element then A is an interval.

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Theorem 3-14. A set of real numbers with more than one element is connected if and only if it is an interval.

Proof (continued). Now suppose that A is an interval (all intervals contain at least two elements). ASSUME A is not connected. Then there are U and V disjoint open sets such that (i) $A \subset U \cup V$ and (ii) $U \cap A \neq \varnothing$ and $V \cap A \neq \varnothing$. Since U is an open set then, by Theorem 3-5, there is a countable collection of disjoint open intervals $\{U_i \mid i = 1, 2, \ldots\}$ with $U = \cup_{i=1}^{\infty} U_i$. Similarly, there is a countable collection of disjoint open intervals $\{V_i \mid i = 1, 2, \ldots\}$ with $V = \cup_{i=1}^{\infty} V_i$. Suppose $a \in U \cap A$ and $b \in V \cap A$. Then $a \in U_j$ for some U_j, and $b \in V_k$ for some V_k. Without loss of generality, suppose $a < b$. Let $V_k = (\alpha, \beta)$. See Kirkwood's Figure 3-2 below.

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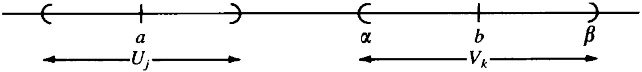

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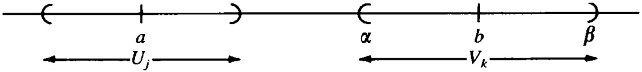

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Proof (continued). Next, we show that α is in neither U nor V. First, if α were in U then (since U is open) there would be $\delta > 0$ such that $(\alpha - \delta, \alpha + \delta) \subset U$. But since $V_k = (\alpha, \beta)$ then we would have $(\alpha - \delta, \alpha + \delta) \cap V_k \neq \varnothing$, from which we would have $U \cap V \neq \varnothing$; this contradicts the disjointness of U and V. So $\alpha \notin U$. Similarly if $\alpha \in V$ then, because $\alpha \notin V_k$, there must be some V_ℓ with $\ell \neq k$ containing α and, since V_ℓ is open, we can conclude that $V_k \cap V_\ell \neq \varnothing$. The contradicts the disjointness of the V_i, so $\alpha \notin V$.

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